TYPE A BLOCKS OF SUPER CATEGORY $\mathcal{O}$

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ABSTRACT. We show that every block of category $\mathcal{O}$ for the general linear Lie superalgebra $\mathfrak{gl}_{m|n}(\Bbbk)$ is equivalent to some corresponding block of category $\mathcal{O}$ for the queer Lie superalgebra $\mathfrak{q}_{m+n}(\Bbbk)$. This implies the truth of the Kazhdan-Lusztig conjecture for the so-called type A blocks of category $\mathcal{O}$ for the queer Lie superalgebra as formulated by Cheng, Kwon and Wang.

1. INTRODUCTION

In this article, we study the analog of the BGG category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\Bbbk)$. Recent work of Chen [C] has reduced most questions about this category just to the study of three particular types of block, which we refer to here as the type A, type B and type C blocks. Type B blocks (which correspond to integral weights) were investigated already by the first author in [B2], leading to a Kazhdan-Lusztig conjecture for characters of irreducibles in such blocks in terms of certain canonical bases for the quantum group of type B_∞. In [CKW], Cheng, Kwon and Wang formulated analogous conjectures for the type A blocks (defined below) and the type C blocks (which correspond to half-integral weights) in terms of canonical bases of quantum groups of types A_∞ and C_∞, respectively.

The main goal of the article is to prove the Cheng-Kwon-Wang conjecture for type A blocks ([CKW, Conjecture 5.14]). To do this, we use some tools from higher representation theory to establish an equivalence of categories between the type A blocks of category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\Bbbk)$ and integral blocks of category $\mathcal{O}$ for a general linear Lie superalgebra. This reduces the Cheng-Kwon-Wang conjecture for type A blocks to the Kazhdan-Lusztig conjecture of [B1], which was proved already in [CLW, BLW].

Regarding the types B and C conjectures, Tsuchioka discovered in 2010 that the type B canonical bases considered in [B2] fail to satisfy appropriate positivity properties, so that the conjecture from [B2] is certainly false. Moreover, after the first version of [CKW] appeared, Tsuchioka pointed out similar issues with the type C canonical bases studied in [CKW], so that the Cheng-Kwon-Wang conjecture for type C blocks as formulated in the first version of their article ([CKW, Conjecture 5.10]) also seems likely to be incorrect.

In fact, the techniques developed in this article can be applied also to the study of the type C blocks. This will be spelled out in a sequel to this paper. In this sequel, we prove a modified version of the Cheng-Kwon-Wang conjecture for type C blocks: one needs to replace Lusztig's canonical basis with Webster's "orthodox basis" arising from the indecomposable projective modules of the tensor product algebras of [W, §4]. This modified conjecture was proposed independently by Cheng, Kwon and Wang in a revision of their article ([CKW, Conjecture 5.12]). It is not as satisfactory as the

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situation for type A blocks, since there is no elementary algorithm to compute Webster's basis explicitly (unlike the canonical basis). Also in the sequel, we will prove [CKW, Conjecture 5.13], and settle [CKW, Question 5.1] by identifying the category of finite-dimensional half-integer weight representations of $\mathfrak{q}_n(\Bbbk)$ with a previously known highest weight category (as suggested by [CK, Remark 6.7]).

There is more to be said about type B blocks too; in fact, these are the most intriguing of all. Whereas the types A and C blocks carry the additional structure of tensor product categorifications in the sense of [LW, BLW] for the infinite rank Kac-Moody algebras of types A_∞ and C_∞, respectively, the type B blocks produce an example of a tensor product categorification of *odd* type B_∞, i.e. one needs a *super* Kac-Moody 2-category in the sense of [BE2]. This will be developed in subsequent work by the second author.

In the remainder of the introduction, we are going to formulate our main result for type A blocks in more detail. To do this, we first briefly recall some basic notions of superalgebra. Let $\Bbbk$ be a ground field which is algebraically closed of characteristic zero, and fix a choice of $\sqrt{-1} \in \Bbbk$. We adopt the language of [BE1, Definition 1.1]:

- A *supercategory* is a category enriched in the symmetric monoidal category of vector superspaces, i.e. the category of $\mathbb{Z}/2$-graded vector spaces over $\Bbbk$ with morphisms that are parity-preserving linear maps.
- Any morphism in a supercategory decomposes uniquely into an even and an odd morphism as $f = f_{\bar{0}} + f_{\bar{1}}$. A *superfunctor* between supercategories means a $\Bbbk$-linear functor which preserves the parities of morphisms.
- For superfunctors $F, G : \mathcal{C} \to \mathcal{D}$, a *supernatural transformation* $\eta : F \Rightarrow G$ is a family of morphisms $\eta_M = \eta_{M,\bar{0}} + \eta_{M,\bar{1}} : FM \to GM$ for each $M \in \operatorname{ob}\mathcal{C}$, such that $\eta_{N,p} \circ Ff = (-1)^{|f|p} Gf \circ \eta_{M,p}$ for every homogeneous morphism $f : M \to N$ in $\mathcal{C}$ and each $p \in \mathbb{Z}/2$.

For any supercategory $\mathcal{C}$, the *Clifford twist* $\mathcal{C}^{\mathtt{CT}}$ is the supercategory whose objects are pairs (X, ϕ) for $X \in \operatorname{ob}\mathcal{C}$ and $\phi \in \operatorname{End}_{\mathcal{C}}(X)_{\bar{1}}$ with $\phi^2 = \mathrm{id}$, and whose morphisms $f : (X, \phi) \to (X', \phi')$ are morphisms $f : X \to X'$ in $\mathcal{C}$ such that $f_p \circ \phi = (-1)^p \phi' \circ f_p$ for each $p \in \mathbb{Z}/2$. One can also take Clifford twists of superfunctors and supernatural transformations (details omitted), so that $\mathtt{CT}$ is actually a 2-superfunctor from the 2-supercategory of supercategories to itself in the sense of [BE1, Definition 2.2]. The following basic lemma is a variation on [KKT, Lemma 2.3].

Lemma. *Suppose $\mathcal{C}$ is a supercategory such that*

- *$\mathcal{C}$ is additive;*
- *$\mathcal{C}$ is Π-complete, i.e. every object of $\mathcal{C}$ is the target of an odd isomorphism;*
- *all even idempotents split.*

Then the supercategories $\mathcal{C}$ and $(\mathcal{C}^{\mathtt{CT}})^{\mathtt{CT}}$ are superequivalent.

For example, suppose that A is a locally unital superalgebra, i.e. an associative superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ equipped with a distinguished collection $\{1_x \mid x \in X\}$ of mutually orthogonal even idempotents such that $A = \bigoplus_{x,y \in X} 1_y A 1_x$. Then there is a supercategory A-smod consisting of finite-dimensional left A-supermodules M which are locally unital in the sense that $M = \bigoplus_{x \in X} 1_x M$. Even morphisms in A-smod are parity-preserving linear maps such that $f(av) = af(v)$ for all $a \in A, v \in M$; odd morphisms are parity-reversing linear maps such that $f(av) = (-1)^{|a|} af(v)$ for homogeneous a. There is an obvious isomorphism between the Clifford twist $A\text{-smod}^{\mathtt{CT}}$ of this supercategory and the supercategory $A \otimes C_1\text{-smod}$, where C_1 denotes the rank one Clifford superalgebra generated by an odd involution c, and $A \otimes C_1$ is the usual braided tensor product of superalgebras. Hence, $(A\text{-smod}^{\mathtt{CT}})^{\mathtt{CT}}$ is isomorphic to $A \otimes C_2\text{-smod}$ where $C_2 := C_1 \otimes C_1$ is the rank two Clifford superalgebra generated by $c_1 := c \otimes 1$ and $c_2 := 1 \otimes c$. In this situation, the above lemma is obvious as $A \otimes C_2$ is isomorphic to the matrix superalgebra $M_{1|1}(A)$, which is Morita superequivalent to A.

Now fix $n \geq 1$ and let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be the Lie superalgebra $\mathfrak{q}_n(\Bbbk)$. Recall this is the subalgebra of the general linear Lie superalgebra $\mathfrak{gl}_{n|n}(\Bbbk)$ consisting of all matrices of block form

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right). \tag{1.1}$$

Let $\mathfrak{b}$ (resp. $\mathfrak{h}$) be the standard Borel (resp. Cartan) subalgebra of $\mathfrak{g}$ consisting of all matrices (1.1) in which A and B are upper triangular (resp. diagonal). Let $\mathfrak{t} := \mathfrak{h}_{\bar{0}}$. We let $\delta_1, \dots, \delta_n$ be the basis for $\mathfrak{t}^*$ such that δ_i picks out the ith diagonal entry of the matrix A. Fix also a sign sequence $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ with each $\sigma_r \in \{\pm\}$, and a scalar $z \in \Bbbk$ such that $2z \notin \mathbb{Z}$. **We stress that all of our subsequent notation depends implicitly on these choices.**

It will be convenient to index certain weights in $\mathfrak{t}^*$ by the set $\mathbf{B} := \mathbb{Z}^n$ via the following *weight dictionary*: for $\boldsymbol{b} = (b_1, \dots, b_n) \in \mathbf{B}$ let

$$\lambda_{\boldsymbol{b}} := \sum_{r=1}^{n} \lambda_{\boldsymbol{b},r}\delta_r \quad \text{where} \quad \lambda_{\boldsymbol{b},r} := \sigma_r(z + b_r). \tag{1.2}$$

We let $s\mathcal{O}$ be the category of all $\mathfrak{g}$-supermodules M satisfying the following properties:

- M is finitely generated as a $\mathfrak{g}$-supermodule;
- M is locally finite-dimensional over $\mathfrak{b}$;
- M is semisimple over $\mathfrak{t}$ with all weights of the form $\lambda_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$.

Morphisms in $s\mathcal{O}$ are arbitrary (not necessarily even) $\mathfrak{g}$-supermodule homomorphisms, so that it is a supercategory. It also admits a parity switching functor Π. The *type A blocks* mentioned earlier are the blocks of $s\mathcal{O}$ for all possible choices of $\boldsymbol{\sigma}$ and z.

For each $\boldsymbol{b} \in \mathbf{B}$, there is an irreducible supermodule $L(\boldsymbol{b}) \in \operatorname{ob} s\mathcal{O}$ of highest weight $\lambda_{\boldsymbol{b}}$. Note the highest weight space of $L(\boldsymbol{b})$ is not one-dimensional: it is some sort of irreducible Clifford supermodule over the Cartan subalgebra $\mathfrak{h}$. Every irreducible supermodule in $s\mathcal{O}$ is isomorphic to $L(\boldsymbol{b})$ for a unique $\boldsymbol{b} \in \mathbf{B}$ via a homogeneous (but not necessarily even) isomorphism. If n is odd, $L(\boldsymbol{b})$ is of type $\mathtt{Q}$, i.e., $L(\boldsymbol{b})$ is evenly isomorphic to its parity flip $\Pi L(\boldsymbol{b})$. When n is even, the irreducible $L(\boldsymbol{b})$ is of type $\mathtt{M}$, and we should explain how to distinguish it from its parity flip. For each $i \in \mathbb{Z}$, we fix a choice $\sqrt{z+i}$ of a square root of $z + i$, then set $\sqrt{-(z+i)} := (-1)^i\sqrt{-1}\sqrt{z+i}$. The key point about this is that

$$\sqrt{-(z+i)}\sqrt{-(z+i+1)} = \sqrt{z+i}\sqrt{z+i+1} \tag{1.3}$$

for each $i \in \mathbb{Z}$. Let $d'_r \in \mathfrak{g}_{\bar{1}}$ be the matrix of the form (1.1) such that $A = 0$ and B is the rth diagonal matrix unit. Then, for even n, we assume that $L(\boldsymbol{b})$ is chosen so that $d'_1 \cdots d'_n$ acts on any even highest weight vector by the scalar $(\sqrt{-1})^{n/2}\sqrt{\lambda_{\boldsymbol{b},1}} \cdots \sqrt{\lambda_{\boldsymbol{b},n}}$. This determines $L(\boldsymbol{b})$ uniquely up to even isomorphism.

Turning our attention to the category on the other side of our main equivalence, let $\mathfrak{g}'$ be the general linear Lie superalgebra consisting of $n \times n$ matrices under the supercommutator, with $\mathbb{Z}/2$-grading defined by declaring that the rs-matrix unit is even if $\sigma_r = \sigma_s$ and odd if $\sigma_r \neq \sigma_s$. Let $\mathfrak{b}'$ (resp. $\mathfrak{t}'$) be the standard Borel (resp. Cartan) subalgebra consisting of upper triangular (resp. diagonal) matrices in $\mathfrak{g}'$. As before, we let $\delta'_1, \dots, \delta'_n$ be the basis for $(\mathfrak{t}')^*$ defined by the diagonal coordinate functions. We introduce another *weight dictionary* (which in this setting is some "signed ρ-shift"): for $\boldsymbol{b} \in \mathbf{B}$, let

$$\lambda'_{\boldsymbol{b}} := \sum_{r=1}^{n} \lambda'_{\boldsymbol{b},r}\delta'_r \quad \text{where} \quad \lambda'_{\boldsymbol{b},r} := \sigma_r\left(b_r + \sigma_1 1 + \cdots + \sigma_{r-1}1 + \tfrac{1}{2}(\sigma_r 1 - 1)\right). \tag{1.4}$$

Let $s\mathcal{O}'$ be the supercategory of $\mathfrak{g}'$-supermodules M' such that

- M' is finitely generated as a $\mathfrak{g}'$-supermodule;
- M' is locally finite-dimensional over $\mathfrak{b}'$;
- M' is semisimple over $\mathfrak{t}'$ with all weights of the form $\lambda'_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$.

Note $s\mathcal{O}'$ is the sum of all of the blocks of the usual category $\mathcal{O}$ for $\mathfrak{g}'$ corresponding to integral weights of $\mathfrak{t}'$. For each $\boldsymbol{b} \in \mathbf{B}$, there is a unique (up to even isomorphism) irreducible supermodule $L'(\boldsymbol{b}) \in \operatorname{ob} s\mathcal{O}'$ generated by a homogeneous highest weight vector of weight $\lambda'_{\boldsymbol{b}}$ and parity $\sum_{\sigma_r=-} \lambda'_{\boldsymbol{b},r} \pmod 2$.

Main Theorem. *If n is even then there is a superequivalence $\mathbb{E} : s\mathcal{O} \to s\mathcal{O}'$ such that $\mathbb{E}L(\boldsymbol{b})$ is evenly isomorphic to $L'(\boldsymbol{b})$ for each $\boldsymbol{b} \in \mathbf{B}$. If n is odd then there is a superequivalence $\mathbb{E} : s\mathcal{O} \to (s\mathcal{O}')^{\mathtt{CT}}$ such that $\mathbb{E}L(\boldsymbol{b})$ is evenly isomorphic to $(L'(\boldsymbol{b}) \oplus \Pi L'(\boldsymbol{b}), \phi)$ for either of the two choices of odd involution ϕ.*

If $\mathcal{C}$ is any $\Bbbk$-linear category, we let $\mathcal{C} \oplus \Pi\mathcal{C}$ be the supercategory whose objects are formal direct sums $V_1 \oplus \Pi V_2$ for $V_1, V_2 \in \operatorname{ob}\mathcal{C}$, with morphisms $V_1 \oplus \Pi V_2 \to W_1 \oplus \Pi W_2$ being matrices of the form $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ for $f_{ij} \in \operatorname{Hom}_{\mathcal{C}}(V_j, W_i)$. The $\mathbb{Z}/2$-grading is defined so that $f_{\bar 0} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}$ and $f_{\bar 1} = \begin{pmatrix} 0 & f_{12} \\ f_{21} & 0 \end{pmatrix}$. For example, if $\mathcal{C}$ is the category A-mod of finite-dimensional locally unital modules over some locally unital algebra A, then $\mathcal{C} \oplus \Pi\mathcal{C}$ may be identified with the category A-smod, viewing A as a purely even superalgebra.

It was noticed originally in [B1] that the category $s\mathcal{O}'$ can be decomposed in this way: let $\mathcal{O}'$ be full subcategory of $s\mathcal{O}'$ consisting of all $\mathfrak{g}'$-supermodules whose $\lambda'_{\boldsymbol{b}}$-weight space is concentrated in parity $\sum_{\sigma_r=-} \lambda'_{\boldsymbol{b},r} \pmod 2$ for each $\boldsymbol{b} \in \mathbf{B}$; obviously, there are no non-zero odd morphisms between objects of $\mathcal{O}'$. Then $s\mathcal{O}'$ decomposes as $s\mathcal{O}' = \mathcal{O}' \oplus \Pi\mathcal{O}'$. Moreover, $\mathcal{O}'$ is a highest weight category with irreducible objects $\{L'(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$ indexed by the set $\mathbf{B}$ as above. In fact, $\mathcal{O}'$ equivalent to A-mod for a locally unital algebra A such that the left ideals $A1_x$ and right ideals 1_xA are finite-dimensional for all distinguished idempotents $1_x \in A$. Although not needed here, the results of [BLW] imply further that the algebra A may be equipped with a $\mathbb{Z}$-grading making it into a (locally unital) *Koszul algebra*; this leads to the definition of a graded analog of the category $\mathcal{O}'$ similar in spirit to Soergel's graded lift of classical category $\mathcal{O}$ as in e.g. [BGS].

Combining these remarks with our Main Theorem, we deduce:

- For even n, the category $s\mathcal{O}$ decomposes as $s\mathcal{O} = \mathcal{O} \oplus \Pi\mathcal{O}$, where $\mathcal{O}$ is the Serre subcategory generated by the irreducible supermodules $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$ introduced above (but *not* their parity flips). Moreover, $\mathcal{O}$ is equivalent to $\mathcal{O}'$, hence, to the category A-mod where A is the Koszul algebra just introduced.

- For odd n, $s\mathcal{O}$ is superequivalent to $A \otimes C_1$-smod, viewing A as a purely even superalgebra. This implies that the underlying category $\underline{s\mathcal{O}}$ consisting of the same objects as $s\mathcal{O}$ but only its even morphisms is equivalent to A-mod, hence, to $\mathcal{O}'$.

As already mentioned, the Kazhdan-Lusztig conjecture for $s\mathcal{O}$ formulated in [CKW] follows immediately from this discussion together with the Kazhdan-Lusztig conjecture for $\mathcal{O}'$ proved in [CLW, BLW].

There is also a parabolic analog of our Main Theorem. Let $\nu = (\nu_1, \dots, \nu_l)$ be a composition of n with $\sigma_r = \sigma_s$ for all $\nu_1 + \cdots + \nu_{k-1} + 1 \leq r < s \leq \nu_1 + \cdots + \nu_k$ and $k = 1, \dots, l$. Let $\mathfrak{p}_\nu$ be the corresponding standard parabolic subalgebra of $\mathfrak{g}$, i.e. the matrices A and B in (1.1) are block upper triangular with diagonal blocks of shape ν. Let $s\mathcal{O}_\nu$ be the corresponding parabolic analog of the category $s\mathcal{O}$, i.e. it is the full subcategory of $s\mathcal{O}$ consisting of all supermodules that are locally finite-dimensional over $\mathfrak{p}_\nu$. Similarly, there is a standard parabolic subalgebra $\mathfrak{p}'_\nu$ of $\mathfrak{g}'$ consisting of block upper triangular matrices of shape ν, and we let $s\mathcal{O}'_\nu$ be the analogously defined parabolic subcategory of $s\mathcal{O}'$. Various special cases of the following corollary for maximal parabolics/two-part compositions ν were known before; see [C, §4] and [CC].

Corollary. *If n is even then $s\mathcal{O}_\nu$ is superequivalent to $s\mathcal{O}'_\nu$. If n is odd then $s\mathcal{O}_\nu$ is superequivalent to $(s\mathcal{O}'_\nu)^{\mathtt{CT}}$.*

Proof. This follows from our Main Theorem on observing that $s\mathcal{O}_\nu$ and $s\mathcal{O}'_\nu$ may be defined equivalently as the Serre subcategories of $s\mathcal{O}$ and $s\mathcal{O}'$ generated by the irreducible supermodules $\{L(\boldsymbol{b}), \Pi L(\boldsymbol{b})\}$ and $\{L'(\boldsymbol{b}), \Pi L'(\boldsymbol{b})\}$, respectively, for $\boldsymbol{b} \in \mathbf{B}$ such that the following hold for $r \notin \{\nu_1, \nu_1 + \nu_2, \dots, \nu_1 + \cdots + \nu_l\}$:

- if $\sigma_r = +$ then $b_r > b_{r+1}$;
- if $\sigma_r = -$ then $b_r < b_{r+1}$.

(This assertion is a well-known consequence of the construction of parabolic Verma supermodules in $s\mathcal{O}$ and $s\mathcal{O}'$, respectively; see e.g. [M].) □

In order to prove the Main Theorem, we will exploit the following powerful theorem established in [BLW]: the category $\mathcal{O}'$ defined above is the unique (up to strongly equivariant equivalence) $\mathfrak{sl}_\infty$-tensor product categorification of the module

$$V^{\otimes \boldsymbol{\sigma}} := V^{\sigma_1} \otimes \cdots \otimes V^{\sigma_n},$$

where V^+ denotes the natural $\mathfrak{sl}_\infty$-module and V^- denotes its dual. Hence, for even n, it suffices to show that the category $s\mathcal{O}$ decomposes as $\mathcal{O} \oplus \Pi\mathcal{O}$ for some $\mathfrak{sl}_\infty$-tensor product categorification $\mathcal{O}$ of $V^{\otimes \boldsymbol{\sigma}}$. For odd n, we show instead that $s\mathcal{O}^{\mathtt{CT}}$ decomposes as $\mathcal{O} \oplus \Pi\mathcal{O}$ for some $\mathfrak{sl}_\infty$-tensor product categorification $\mathcal{O}$ of $V^{\otimes \boldsymbol{\sigma}}$, hence, $s\mathcal{O}^{\mathtt{CT}}$ is superequivalent to $s\mathcal{O}'$. The Main Theorem then follows on taking Clifford twists, using also the lemma formulated above. In both the even and odd cases, our argument relies crucially also on an application of the main result of [KKT].

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2. Verma supermodules

We continue with $n \geq 1$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \{\pm 1\}^n$, and $z \in \Bbbk$ with $2z \notin \mathbb{Z}$. Let $m := \lceil n/2 \rceil$, so that $n = 2m$ or $2m - 1$. Also set

$$I := \mathbb{Z}, \qquad J := \left\{ \pm\sqrt{z+i}\sqrt{z+i+1} \;\middle|\; i \in I \right\}, \tag{2.1}$$

where the square roots are as chosen in the introduction.

In this paragraph, we work with the Lie superalgebra $\widehat{\mathfrak{g}} := \mathfrak{gl}_{2m|2m}(\Bbbk)$ in order to introduce some coordinates. Let $\widehat{U}$ be the natural $\widehat{\mathfrak{g}}$-supermodule with standard basis $u_1, \dots, u_{4m}$. Write $x_{r,s}$ for the rs-matrix unit in $\widehat{\mathfrak{g}}$, so $x_{r,s}u_t = \delta_{s,t}u_r$. We denote the odd basis vectors $u_{2m+1}, \dots, u_{4m}$ instead by $u_1', \dots, u_{2m}'$. For $1 \leq r, s \leq 2m$, we set

$$e_{r,s} := x_{r,s} + x_{2m+r,2m+s}, \qquad e_{r,s}' := x_{r,2m+s} + x_{2m+r,s}, \tag{2.2}$$

$$f_{r,s} := x_{r,s} - x_{2m+r,2m+s}, \qquad f_{r,s}' := x_{r,2m+s} - x_{2m+r,s}. \tag{2.3}$$

Also let

$$d_r := e_{r,r}, \qquad d_r' := e_{r,r}'. \tag{2.4}$$

Then we have that

$$e_{r,s}u_t = \delta_{s,t}u_r, \qquad e_{r,s}u_t' = \delta_{s,t}u_r', \qquad e_{r,s}'u_t = \delta_{s,t}u_r', \qquad e_{r,s}'u_t' = \delta_{s,t}u_r, \tag{2.5}$$

$$f_{r,s}u_t = \delta_{s,t}u_r, \qquad f_{r,s}u_t' = -\delta_{s,t}u_r', \qquad f_{r,s}'u_t = -\delta_{s,t}u_r', \qquad f_{r,s}'u_t' = \delta_{s,t}u_r. \tag{2.6}$$

Finally let $\widehat{U}^*$ be the dual supermodule to $\widehat{U}$, with basis $\phi_1, \dots, \phi_{2m}, \phi_1', \dots, \phi_{2m}'$ that is dual to the basis $u_1, \dots, u_{2m}, u_1', \dots, u_{2m}'$. We have that

$$e_{r,s}\phi_t = -\delta_{r,t}\phi_s, \qquad e_{r,s}\phi_t' = -\delta_{r,t}\phi_s', \qquad e_{r,s}'\phi_t = -\delta_{r,t}\phi_s', \qquad e_{r,s}'\phi_t' = \delta_{r,t}\phi_s, \tag{2.7}$$

$$f_{r,s}\phi_t = -\delta_{r,t}\phi_s, \qquad f_{r,s}\phi_t' = \delta_{r,t}\phi_s', \qquad f_{r,s}'\phi_t = -\delta_{r,t}\phi_s', \qquad f_{r,s}'\phi_t' = -\delta_{r,t}\phi_s. \tag{2.8}$$

When n is even, we continue with $\mathfrak{g}, \mathfrak{b}$ and $\mathfrak{h}$ as in the introduction, so $\mathfrak{g}$ is the subalgebra of $\widehat{\mathfrak{g}}$ spanned by $\{e_{r,s}, e_{r,s}' | 1 \leq r, s \leq n\}$, while $\mathfrak{h}$ has basis $\{d_r, d_r' | 1 \leq r \leq 2m\}$. However, when n is odd, it is convenient to change some of this notation. The point of doing this is to unify our treatment of even and odd n as much as possible in the remainder of the article. So, if n is odd, we henceforth *redefine* $\mathfrak{g}, \mathfrak{b}$ and $\mathfrak{h}$ as follows:

- $\mathfrak{g}$ denotes the Lie superalgebra $\mathfrak{q}_n(\Bbbk) \oplus \mathfrak{q}_1(\Bbbk)$, which we identify with the subalgebra of $\widehat{\mathfrak{g}}$ spanned by $\{e_{r,s}, e_{r,s}' \mid 1 \leq r, s \leq n\} \sqcup \{d_{2m}, d_{2m}'\}$.
- $\mathfrak{b}$ is the Borel subalgebra spanned by $\{e_{r,s}, e_{r,s}' \mid 1 \leq r \leq s \leq n\} \sqcup \{d_{2m}, d_{2m}'\}$;
- $\mathfrak{h}$ is the Cartan subalgebra spanned by $\{d_r, d_r' \mid 1 \leq r \leq 2m\}$.

In both the even and the odd cases, the subspaces $U \subseteq \widehat{U}$ and $U^* \subseteq \widehat{U}^*$ spanned by $u_1, \dots, u_n, u_1', \dots, u_n'$ and $\phi_1, \dots, \phi_n, \phi_1', \dots, \phi_n'$, respectively, may be viewed as $\mathfrak{g}$-supermodules. Also set $\mathfrak{t} := \mathfrak{h}_{\bar{0}}$ and let $\delta_1, \dots, \delta_{2m}$ be the basis for $\mathfrak{t}^*$ that is dual to the basis $d_1, \dots, d_{2m}$ for $\mathfrak{t}$.

For $\boldsymbol{b} \in \mathbf{B}$, we define $\lambda_{\boldsymbol{b}}$ according to (1.2) if n is even, but redefine it in the odd case as

$$\lambda_{\boldsymbol{b}} := \sum_{r=1}^{n} \lambda_{\boldsymbol{b},r}\delta_r + \delta_{2m} \quad \text{where} \quad \lambda_{\boldsymbol{b},r} := \sigma_r(z + b_r). \tag{2.9}$$

We also introduce the tuple $\boldsymbol{d}_r \in \mathbf{B}$ which has 1 as its rth entry and 0 in all other places, so that

$$\lambda_{\boldsymbol{b}} \pm \delta_r = \lambda_{\boldsymbol{b}\pm\sigma_r \boldsymbol{d}_r}. \tag{2.10}$$

Then we define $s\mathcal{O}$ exactly as we did in the introduction but using the current choices for $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ and $\lambda_{\boldsymbol{b}}$. This is exactly the same category as in the introduction when n is even, but when n is odd our new version of $s\mathcal{O}$ is superequivalent to the Clifford twist $s\mathcal{O}^{\mathtt{CT}}$ of the supercategory from the introduction. Indeed, if M is a supermodule in our new $s\mathcal{O}$, the restriction of M to the subalgebra $\mathfrak{q}_n(\Bbbk)$, equipped with the odd involution defined by the action of d'_{2m}, gives an object of the Clifford twist of the supercategory from before.

We proceed to define some irreducible $\mathfrak{h}$-supermodules $\{V(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$. Let C_2 be the rank 2 Clifford superalgebra with odd generators c_1, c_2 subject to the relations $c_1^2 = c_2^2 = 1, c_1c_2 = -c_2c_1$. Let V be the irreducible C_2-supermodule on basis v, v' with v even and v' odd, and action defined by

$$c_1 v = v', \quad c_1 v' = v, \quad c_2 v = \sqrt{-1}v', \quad c_2 v' = -\sqrt{-1}v.$$

Then, for $\boldsymbol{b} \in \mathbf{B}$, we set $V(\boldsymbol{b}) := V^{\otimes m}$. For $1 \le r \le n$, we let d_r act by the scalar $\lambda_{\boldsymbol{b},r}$ and d'_r act by left multiplication by $\sqrt{\lambda_{\boldsymbol{b},r}}\ \mathrm{id}^{\otimes(s-1)} \otimes c_{r+1-2s} \otimes \mathrm{id}^{(m-s)}$ where $s := \lfloor r/2 \rfloor$ (and we are using the usual superalgebra sign rules). In the odd case, we also need to define the actions of d_{2m} and d'_{2m}: these are the identity and the odd involution $\mathrm{id}^{\otimes(m-1)} \otimes c_2$, respectively. In all cases, $V(\boldsymbol{b})$ is an irreducible $\mathfrak{h}$-supermodule of type $\mathtt{M}$, and its $\mathfrak{t}$-weight is $\lambda_{\boldsymbol{b}}$. Moreover, by construction, $d'_1 \cdots d'_{2m}$ acts on any even (resp. odd) vector in $V(\boldsymbol{b})$ as $c_{\boldsymbol{b}}$ (resp. $-c_{\boldsymbol{b}}$), where

$$c_{\boldsymbol{b}} := (\sqrt{-1})^m \sqrt{\lambda_{\boldsymbol{b},1}} \cdots \sqrt{\lambda_{\boldsymbol{b},n}}. \tag{2.11}$$

The signs here distinguish $V(\boldsymbol{b})$ from its parity flip.

Lemma 2.1. *For $\boldsymbol{b} \in \mathbf{B}$, any $\mathfrak{h}$-supermodule that is semisimple of weight $\lambda_{\boldsymbol{b}}$ over $\mathfrak{t}$ decomposes as a direct sum of copies of the supermodules $V(\boldsymbol{b})$ and $\Pi V(\boldsymbol{b})$.*

Proof. We can identify $\mathfrak{h}$-supermodules that are semisimple of weight $\lambda_{\boldsymbol{b}}$ over $\mathfrak{t}$ with supermodules over the Clifford superalgebra $C_{2m} := C_2^{\otimes m}$, so that d'_r $(r = 1, \dots, n)$ acts in the same way as $\sqrt{\lambda_{\boldsymbol{b},r}}\ \mathrm{id}^{\otimes(s-1)} \otimes c_{r+1-2s} \otimes \mathrm{id}^{(m-s)}$ where $s := \lfloor r/2 \rfloor$, and in the odd case d'_{2m} acts as $\mathrm{id}^{\otimes(m-1)} \otimes c_2$. The lemma then follows since C_{2m} is simple, indeed, it is isomorphic to the matrix superalgebra $M_{2^{n-1}|2^{n-1}}(\Bbbk)$. □

Let $\underline{s\mathcal{O}}$ denote the *underlying category* consisting of the same objects as $s\mathcal{O}$ but only the even morphisms. This is obviously an Abelian category. In order to parametrize its irreducible objects explicitly, we introduce the *Verma supermodule* $M(\boldsymbol{b})$ for $\boldsymbol{b} \in \mathbf{B}$ by setting

$$M(\boldsymbol{b}) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V(\boldsymbol{b}), \tag{2.12}$$

where we are viewing $V(\boldsymbol{b})$ as a $\mathfrak{b}$-supermodule by inflating along the surjection $\mathfrak{b} \twoheadrightarrow \mathfrak{h}$. The weight $\lambda_{\boldsymbol{b}}$ is the highest weight of $M(\boldsymbol{b})$ in the usual *dominance order* on $\mathfrak{t}^*$, i.e. $\lambda \le \mu$ if and only if $\mu - \lambda \in \bigoplus_{r=1}^{n-1} \mathbb{N}(\delta_r - \delta_{r+1})$. Note also that we can distinguish $M(\boldsymbol{b})$ from its parity flip in the same way as for $V(\boldsymbol{b})$: the element $d'_1 \cdots d'_{2m}$ acts on any even (resp. odd) vector in the highest weight space $M(\boldsymbol{b})_{\lambda_{\boldsymbol{b}}}$ as the scalar $c_{\boldsymbol{b}}$ (resp. $-c_{\boldsymbol{b}}$).

As usual, the Verma supermodule $M(\boldsymbol{b})$ has a unique irreducible quotient denoted $L(\boldsymbol{b})$. Thus, $L(\boldsymbol{b})$ is an irreducible $\mathfrak{g}$-supermodule of highest weight $\lambda_{\boldsymbol{b}}$, and the action of $d_1' \cdots d_{2m}'$ on its highest weight space distinguishes it from its parity flip. The irreducible supermodules $\{L(\boldsymbol{b}), \Pi L(\boldsymbol{b}) \,|\, \boldsymbol{b} \in \mathbf{B}\}$ give a complete set of pairwise inequivalent irreducible supermodules in $\underline{s\mathcal{O}}$. The endomorphism algebras of these objects are all one-dimensional, so they are irreducibles of type $\mathtt{M}$. Moreover, by a standard argument involving restricting to the underlying even Lie algebra as in [B3, Lemma 7.3], we get that $\underline{s\mathcal{O}}$ is a Schurian category in the following sense (cf. [BLW, §2.1]):

Definition 2.2. A *Schurian category* is a $\Bbbk$-linear Abelian category in which all objects have finite length, there are enough projectives and injectives, and the endomorphism algebras of irreducible objects are all one-dimensional.

Let x^T denote the usual transpose of a matrix $x \in \widehat{\mathfrak{g}}$. This induces an antiautomorphism of $\mathfrak{g}$, i.e. we have that $[x, y]^T = [y^T, x^T]$. Given $M \in \operatorname{ob} s\mathcal{O}$, we can view the direct sum $\bigoplus_{\boldsymbol{b} \in \mathbf{B}} M_{\lambda_{\boldsymbol{b}}}^*$ of the linear duals of the weight spaces of M as a $\mathfrak{g}$-supermodule with action defined by $(xf)(v) := f(x^T v)$. Let $M^\star$ be the object of $s\mathcal{O}$ obtained from this by applying also the parity switching functor Π^m. Making the obvious definition on morphisms, this gives us a contravariant superequivalence $\star : s\mathcal{O} \to s\mathcal{O}$. We have incorporated the parity flip into this definition in order to get the following lemma.

Lemma 2.3. *For $\boldsymbol{b} \in \mathbf{B}$, we have that $L(\boldsymbol{b})^\star \cong L(\boldsymbol{b})$ via an even isomorphism.*

Proof. By weight considerations, we either have that $L(\boldsymbol{b})^\star$ is evenly isomorphic to $L(\boldsymbol{b})$ or to $\Pi L(\boldsymbol{b})$. To show that the former holds, take an even highest weight vector $f \in L(\boldsymbol{b})^\star$. We must show that $d_1' \cdots d_{2m}' f = c_{\boldsymbol{b}} f$ (rather than $-c_{\boldsymbol{b}} f$). Remembering the twist by Π^m in our definition of $\star$, there is a highest weight vector $v \in L(\boldsymbol{b})$ of parity $m \pmod 2$ such that $f(v) = 1$. Then we get that

$$(d_1' \cdots d_{2m}' f)(v) = f(d_{2m}' \cdots d_1' v) = (-1)^m f(d_1' \cdots d_{2m}' v) = c_{\boldsymbol{b}} f(v).$$

Hence, $d_1' \cdots d_{2m}' f = c_{\boldsymbol{b}} f$. $\square$

Let $P(\boldsymbol{b})$ be a projective cover of $L(\boldsymbol{b})$ in $\underline{s\mathcal{O}}$. There are even epimorphisms $P(\boldsymbol{b}) \twoheadrightarrow M(\boldsymbol{b}) \twoheadrightarrow L(\boldsymbol{b})$. Applying $\star$, we deduce that there are even monomorphisms $L(\boldsymbol{b}) \hookrightarrow M(\boldsymbol{b})^\star \hookrightarrow P(\boldsymbol{b})^\star$. The supermodule $P(\boldsymbol{b})^\star$ is an injective hull of $L(\boldsymbol{b})$, while $M(\boldsymbol{b})^\star$ is the *dual Verma supermodule*. The following lemma is well known; it follows from central character considerations (e.g. see [CW, Theorem 2.48]) plus the universal property of Verma supermodules.

Lemma 2.4. *Suppose that $\lambda_{\boldsymbol{b}}$ is dominant and typical, by which we mean that the following hold for all $1 \leq r < s \leq n$:*

- *if $\sigma_r = \sigma_s$ then $\lambda_{\boldsymbol{b},r} \geq \lambda_{\boldsymbol{b},s}$;*
- *if $\sigma_r \neq \sigma_s$ then $\lambda_{\boldsymbol{b},r} + \lambda_{\boldsymbol{b},s} \neq 0$.*

Then $M(\boldsymbol{b}) = P(\boldsymbol{b})$.

Let $s\mathcal{O}^\Delta$ be the full subcategory of $s\mathcal{O}$ consisting of all supermodules possessing a Verma flag, i.e. for which there is a filtration $0 = M_0 \subset \cdots \subset M_l = M$ with sections M_k/M_{k-1} that are evenly isomorphic to $M(\boldsymbol{b})$'s or $\Pi M(\boldsymbol{b})$'s for $\boldsymbol{b} \in \mathbf{B}$. Since the classes of all $M(\boldsymbol{b})$ and $\Pi M(\boldsymbol{b})$ are linearly independent in the Grothendieck group of $\underline{s\mathcal{O}}$, the multiplicities $(M : M(\boldsymbol{b}))$ and $(M : \Pi M(\boldsymbol{b}))$ of $M(\boldsymbol{b})$ and $\Pi M(\boldsymbol{b})$ in a Verma flag of M are independent of the particular choice of flag. The following lemma is quite standard.

Lemma 2.5. *For $M \in \operatorname{ob} s\mathcal{O}^{\Delta}$ and $\boldsymbol{b} \in \mathbf{B}$, we have that*

$$(M : M(\boldsymbol{b})) = \dim \operatorname{Hom}_{s\mathcal{O}}(M, M(\boldsymbol{b})^{\star})_{\bar{0}},$$
$$(M : \Pi M(\boldsymbol{b})) = \dim \operatorname{Hom}_{s\mathcal{O}}(M, M(\boldsymbol{b})^{\star})_{\bar{1}}.$$

Also, any direct summand of $M \in \operatorname{ob} s\mathcal{O}^{\Delta}$ possesses a Verma flag.

Proof. The first part of the lemma follows by induction on the length of the Verma flag, using the following two observations: for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}$ we have that

- $\operatorname{Hom}_{s\mathcal{O}}(M(\boldsymbol{a}), M(\boldsymbol{b})^{\star})$ is zero if $\boldsymbol{a} \neq \boldsymbol{b}$, and it is one-dimensional of even parity if $\boldsymbol{a} = \boldsymbol{b}$;
- $\operatorname{Ext}^1_{s\mathcal{O}}(M(\boldsymbol{a}), M(\boldsymbol{b})^{\star}) = 0$.

To check these, for the first one, we use the universal property of $M(\boldsymbol{a})$ to see that $\operatorname{Hom}_{s\mathcal{O}}(M(\boldsymbol{a}), M(\boldsymbol{b})^{\star})$ is zero unless $\lambda_{\boldsymbol{a}} \leq \lambda_{\boldsymbol{b}}$. Similarly, on applying $\star$, it is zero unless $\lambda_{\boldsymbol{b}} \leq \lambda_{\boldsymbol{a}}$. Hence, we may assume that $\boldsymbol{a} = \boldsymbol{b}$. Finally, any non-zero homomorphism $M(\boldsymbol{a}) \to M(\boldsymbol{a})^{\star}$ must send the head to the socle, so $\operatorname{Hom}_{s\mathcal{O}}(M(\boldsymbol{a}), M(\boldsymbol{a})^{\star})$ is evenly isomorphic to $\operatorname{Hom}_{s\mathcal{O}}(L(\boldsymbol{a}), L(\boldsymbol{a})^{\star})$, which is one-dimensional and even thanks to Lemma 2.3. For the second property, we must show that all short exact sequences in $\underline{s\mathcal{O}}$ of the form

$$0 \to M(\boldsymbol{a})^{\star} \to M \to M(\boldsymbol{b}) \to 0 \qquad \text{or} \qquad 0 \to \Pi M(\boldsymbol{a})^{\star} \to M \to M(\boldsymbol{b}) \to 0$$

split. Either $\lambda_{\boldsymbol{a}}$ or $\lambda_{\boldsymbol{b}}$ is a maximal weight of M. In the latter case, using also Lemma 2.1, we can use the universal property of $M(\boldsymbol{b})$ to construct a splitting of $M \twoheadrightarrow M(\boldsymbol{b})$. In the former case, we apply $\star$, the resulting short exact sequence splits as before, and then we dualize again.

The final statement of the lemma may be proved by mimicking the argument for semisimple Lie algebras from [H, §3.2]. □

3. Special projective superfunctors

Next, we investigate the superfunctors $U \otimes -$ and $U^* \otimes -$ defined by tensoring with the $\mathfrak{g}$-supermodules U and U^* introduced in the previous section. They clearly preserve the properties of being finitely generated over $\mathfrak{g}$, locally finite-dimensional over $\mathfrak{b}$, and semisimple over $\mathfrak{t}$. Since the $\mathfrak{t}$-weights of U and U^* are $\delta_1, \dots, \delta_n$ and $-\delta_1, \dots, -\delta_n$, respectively, and using (2.10), we get for each $M \in \operatorname{ob} s\mathcal{O}$ that all weights of $U \otimes M$ and $U^* \otimes M$ are of the form $\lambda_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$. Hence, these superfunctors send objects of $s\mathcal{O}$ to objects of $s\mathcal{O}$, i.e. we have defined

$$sF := U \otimes - : s\mathcal{O} \to s\mathcal{O}, \qquad sE := U^* \otimes - : s\mathcal{O} \to s\mathcal{O}. \tag{3.1}$$

Let

$$\omega := \sum_{r,s=1}^{n} \left(f_{r,s} \otimes e_{s,r} - f'_{r,s} \otimes e'_{s,r} \right) \in U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g}). \tag{3.2}$$

Left multiplication by ω (resp. by $-\omega$) defines a linear map $x_M : U \otimes M \to U \otimes M$ (resp. $x^*_M : U^* \otimes M \to U^* \otimes M$) for each $\mathfrak{g}$-supermodule M. In view of the next lemma, these maps define a pair of even supernatural transformations

$$x : sF \Rightarrow sF, \qquad x^* : sE \Rightarrow sE. \tag{3.3}$$

Lemma 3.1. *The linear maps x_M and x^*_M just defined are even $\mathfrak{g}$-supermodule homomorphisms.*

Proof. This is straightforward to verify directly, but we give a more conceptual argument which better explains the origin of these maps. The odd element

$$f' := \sum_{t=1}^{n} f'_{t,t} \in U(\widehat{\mathfrak{g}}) \tag{3.4}$$

supercommutes with the elements of $U(\mathfrak{g})$. Hence, $f' \otimes 1 \in U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g})$ supercommutes with the image of the comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \subset U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g})$. The odd Casimir tensor

$$\Omega' := \sum_{r,s=1}^{n} \left(e_{r,s} \otimes e'_{s,r} - e'_{r,s} \otimes e_{s,r}\right) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

also supercommutes with the image of Δ. Hence, the even tensor

$$\Omega := \Omega'(f' \otimes 1) = -\sum_{r,s,t=1}^{n} \left(e_{r,s} f'_{t,t} \otimes e'_{s,r} + e'_{r,s} f'_{t,t} \otimes e_{s,r}\right) \in U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g})$$

commutes with the image of Δ. Consequently, left multiplication by Ω defines even $\mathfrak{g}$-supermodule endomorphisms $x_M : U \otimes M \to U \otimes M$ and $x^*_M : U^* \otimes M \to U^* \otimes M$. It remains to observe that these endomorphisms agree with the linear maps defined by left multiplication by ω and $-\omega$, respectively. Indeed, by a calculation using (2.5)–(2.8), the elements $e_{r,s} f'_{t,t}$ and $e'_{r,s} f'_{t,t}$ of $U(\widehat{\mathfrak{g}})$ act on vectors in U (resp. U^*) in the same way as $\delta_{s,t} f'_{r,s}$ and $-\delta_{s,t} f_{r,s}$ (resp. $-\delta_{r,t} f'_{r,s}$ and $\delta_{r,t} f_{r,s}$), respectively. ☐

Lemma 3.2. *Suppose that $\boldsymbol{b} \in \mathbf{B}$ and let $M := M(\boldsymbol{b})$.*

(1) *There is a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = U \otimes M$$

with $M_t/M_{t-1} \cong M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t) \oplus \Pi M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t)$ for each $t = 1, \dots, n$. The endomorphism x_M preserves this filtration, and the induced endomorphism of M_t/M_{t-1} is diagonalizable with exactly two eigenvalues $\pm\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$. Its $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$-eigenspace is evenly isomorphic to $M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t)$, while the other eigenspace is evenly isomorphic to $\Pi M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t)$.

(2) *There is a filtration*

$$0 = M^n \subset \cdots \subset M^1 \subset M^0 = U^* \otimes M$$

with $M^{t-1}/M^t \cong M(\boldsymbol{b} - \sigma_t \boldsymbol{d}_t) \oplus \Pi M(\boldsymbol{b} - \sigma_t \boldsymbol{d}_t)$ for each $t = 1, \dots, n$. The endomorphism x^*_M preserves this filtration, and the induced endomorphism of M^{t-1}/M^t is diagonalizable with exactly two eigenvalues $\pm\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}-1}$. Its $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}-1}$-eigenspace is evenly isomorphic to $M(\boldsymbol{b} - \sigma_t \boldsymbol{d}_t)$, while the other eigenspace is evenly isomorphic to $\Pi M(\boldsymbol{b} - \sigma_t \boldsymbol{d}_t)$.

Proof. (1) The filtration is constructed in [B2, Lemma 4.3.7], as follows. By the tensor identity

$$U \otimes M = U \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V(\boldsymbol{b})) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U \otimes V(\boldsymbol{b})).$$

As a $\mathfrak{b}$-supermodule, U has a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$ in which the section U_t/U_{t-1} is spanned by the images of u_t and u'_t. Let M_t be the submodule of $U \otimes M$ that maps to $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U_t \otimes V(\lambda))$ under this isomorphism.

Now fix $t \in \{1, \dots, n\}$. Let $v_1, \dots, v_k$ be a basis for the even highest weight space $M(\boldsymbol{b})_{\lambda_{\boldsymbol{b}},\bar{0}}$, so that $d'_t v_1, \dots, d'_t v_k$ is a basis for $M(\boldsymbol{b})_{\lambda_{\boldsymbol{b}},\bar{1}}$. The subquotient $M_t/M_{t-1} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U_t/U_{t-1} \otimes V(\boldsymbol{b}))$ is generated by the images of the vectors $\{u_t \otimes v_i, u'_t \otimes v_i, u_t \otimes d'_t v_i, u'_t \otimes d'_t v_i \,|\, i = 1, \dots, k\}$, which by weight considerations span a $\mathfrak{b}$-supermodule isomorphic to $V(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t) \oplus \Pi V(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t)$. Hence,

$$M_t/M_{t-1} \cong M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t) \oplus \Pi M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t).$$

The action of $f_{r,s} \otimes e_{s,r} - f'_{r,s} \otimes e'_{s,r}$ on any of $u_t \otimes v_i, u'_t \otimes v_i, u_t \otimes d'_t v_i$ or $u'_t \otimes d'_t v_i$ is zero unless $r \le s = t$, and if $r < s = t$ then it sends these vectors into M_{t-1}. Therefore, x_M preserves the filtration. Moreover, this argument shows that it acts on the highest weight space of the quotient M_t/M_{t-1} in the same way as $x_t := f_{t,t} \otimes d_t - f'_{t,t} \otimes d'_t$.

Now consider the purely even subspace $S_{i,t}$ of M_t/M_{t-1} with basis given by the images of $u_t \otimes v_i, u'_t \otimes d'_t v_i$. Recalling that d_t acts on v_i and on $d'_t v_i$ by $\lambda_{\boldsymbol{b},t}$, and that $(d'_t)^2 = d_t$, it is straightforward to check that the matrix of the endomorphism x_t of $S_{i,t}$ in the given basis is equal to

$$A := \begin{pmatrix} \lambda_{\boldsymbol{b},t} & \lambda_{\boldsymbol{b},t} \\ 1 & -\lambda_{\boldsymbol{b},t} \end{pmatrix}.$$

Also recall from our construction of $V(\boldsymbol{b})$ that $d'_1 \cdots d'_{2m}$ acts on v_i as the scalar $c_{\boldsymbol{b}}$ from (2.11), and it acts on $d'_t v_i$ as $-c_{\boldsymbol{b}}$. Using this, another calculation shows that $d'_1 \cdots d'_{2m}$ acts on $S_{i,t}$ as the matrix $\frac{c_{\boldsymbol{b}}}{\lambda_{\boldsymbol{b},t}} A$. Similarly, on the purely odd subspace $S'_{i,t}$ with basis given by the images of $u'_t \otimes v_i, u_t \otimes d'_t v_i$, x_t has matrix $-A$ and $d'_1 \cdots d'_{2m}$ has matrix $-\frac{c_{\boldsymbol{b}}}{\lambda_{\boldsymbol{b},t}} A$.

Since the matrix A has eigenvalues $\pm\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$, the calculation made in the previous paragraph implies that x_t is diagonalizable on M_t/M_{t-1} with exactly these eigenvalues. Moreover on any even highest weight vector in its $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$-eigenspace, we get that $d'_1 \cdots d'_{2m}$ acts as

$$\frac{c_{\boldsymbol{b}}}{\lambda_{\boldsymbol{b},t}} \sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1} = c_{\boldsymbol{b}+\sigma_t \boldsymbol{d}_t}.$$

This implies that the $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$-eigenspace is evenly isomorphic to $M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t)$. Similarly, the $-\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$-eigenspace is evenly isomorphic to $\Pi M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t)$.

(2) Similar. □

Corollary 3.3. *For $M \in \operatorname{ob} s\mathcal{O}$, all roots of the minimal polynomials of x_M and x^*_M (computed in the finite dimensional superalgebras $\operatorname{End}_{s\mathcal{O}}(sF\,M)$ and $\operatorname{End}_{s\mathcal{O}}(sE\,M)$) belong to the set J from (2.1).*

Proof. This is immediate from the theorem in case M is a Verma supermodule. We may then deduce that it is true for all irreducibles, hence, for any $M \in \operatorname{ob} s\mathcal{O}$. □

Corollary 3.3 implies that we can decompose

$$sF = \bigoplus_{j \in J} sF_j, \qquad sE = \bigoplus_{j \in J} sE_j, \tag{3.5}$$

where sF_j (resp. sE_j) is the subfunctor of sF (resp. sE) defined by letting $sF_j\,M$ (resp. $sE_j\,M$) be the generalized j-eigenspace of x_M (resp. x^*_M) for each $M \in \operatorname{ob} s\mathcal{O}$. Recall that I denotes $\mathbb{Z}$. For $i \in I$, we define the *i-signature* of $\boldsymbol{b} \in \mathbf{B}$ to be the n-tuple $i\text{-sig}(\boldsymbol{b}) = (i\text{-sig}_1(\boldsymbol{b}), \dots, i\text{-sig}_n(\boldsymbol{b})) \in \{\mathtt{e}, \mathtt{f}, \bullet\}^n$ with

$$i\text{-sig}_t(\boldsymbol{b}) := \begin{cases} \mathtt{f} & \text{if either } \sigma_t = + \text{ and } b_t = i, \text{ or } \sigma_t = - \text{ and } b_t = i+1, \\ \mathtt{e} & \text{if either } \sigma_t = + \text{ and } b_t = i+1, \text{ or } \sigma_t = - \text{ and } b_t = i, \\ \bullet & \text{otherwise.} \end{cases} \tag{3.6}$$

Theorem 3.4. *Given $\boldsymbol{b} \in \mathbf{B}$ and $i \in I$, let $j := \sqrt{z+i}\sqrt{z+i+1}$. Then:*

(1) $sF_j\, M(\boldsymbol{b})$ has a multiplicity-free filtration with sections that are evenly isomorphic to the Verma supermodules

$$\{M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t) \mid \textit{for } 1 \le t \le n \textit{ such that } i\text{-sig}_t(\boldsymbol{b}) = \mathtt{f}\},$$

appearing from bottom to top in order of increasing index t.

(2) $sE_j\, M(\boldsymbol{b})$ has a multiplicity-free filtration with sections that are evenly isomorphic to the Verma supermodules

$$\{M(\boldsymbol{b} - \sigma_t \boldsymbol{d}_t) \mid \textit{for } 1 \le t \le n \textit{ such that } i\text{-sig}_t(\boldsymbol{b}) = \mathtt{e}\},$$

appearing from top to bottom in order of increasing index t.

Proof. (1) It is immediate from Lemma 3.2 that $sF_j\, M(\boldsymbol{b})$ has a multiplicity-free filtration with sections that are evenly isomorphic to the supermodules $M(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t)$ for $t = 1, \dots, n$ such that $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1} = j$. Squaring both sides, this equation implies that $\left(\lambda_{\boldsymbol{b},t} + \frac{1}{2}\right)^2 = \left(z + i + \frac{1}{2}\right)^2$. Hence,

$$\lambda_{\boldsymbol{b},t} = \sigma_t(z + b_t) = -\tfrac{1}{2} \pm \left(z + i + \tfrac{1}{2}\right).$$

We deduce either that $\sigma_t = +$ and $b_t = i$, or $\sigma_t = -$ and $b_t = i+1$. Since we squared our original equation, it remains to check that we do indeed get solutions to that in both cases. This is clear in the case that $\sigma_t = +$, and it follows in the case that $\sigma_t = -$ using also (1.3).

(2) Similar. $\square$

Remark 3.5. Using Theorem 6.4 below, one can show that there are odd supernatural isomorphisms $c : sF_j \overset{\sim}{\Rightarrow} sF_{-j}$ and $c^* : sE_j \overset{\sim}{\Rightarrow} sE_{-j}$ for each $j \in J$. One consequence (which could be checked directly right away) is that there is another version of Theorem 3.4, in which one takes $j := -\sqrt{z+i}\sqrt{z+i+1}$ and replaces the Verma supermodules $M(\boldsymbol{b} \pm \sigma_t \boldsymbol{d}_t)$ in the statement by their parity flips.

The superfunctors sF and sE are both left and right adjoint to each other via some canonical (even) adjunctions. The adjunction making (sE, sF) into an adjoint pair is induced by the linear maps

$$\varepsilon : U^* \otimes U \to \Bbbk,\ \phi \otimes u \mapsto \phi(u), \qquad \eta : \Bbbk \to U \otimes U^*,\ 1 \mapsto \sum_{r=1}^n (u_r \otimes \phi_r + u_r' \otimes \phi_r').$$

Thus, the unit of adunction $c : 1 \Rightarrow sF\, sE$ is defined on supermodule M by the map $c_M : M \xrightarrow{\text{can}} \Bbbk \otimes M \xrightarrow{\eta \otimes \text{id}} U \otimes U^* \otimes M$, and the counit of adjunction $d : sE\, sF \Rightarrow 1$ is defined by $d_M : U^* \otimes U \otimes M \xrightarrow{\varepsilon \otimes \text{id}} \Bbbk \otimes M \xrightarrow{\text{can}} M$. Similarly, the adjunction making (sF, sE) into an adjoint pair is induced by the linear maps

$$U \otimes U^* \to \Bbbk,\ u \otimes \phi \mapsto (-1)^{|\phi||u|}\phi(u), \quad \Bbbk \to U^* \otimes U,\ 1 \mapsto \sum_{r=1}^n (\phi_r \otimes u_r - \phi_r' \otimes u_r').$$

The following lemma implies that these adjunctions restrict to adjunctions making (sF_j, sE_j) and (sE_j, sF_j) into adjoint pairs for each $j \in J$. It follows that all of these superfunctors send projectives to projectives, and they are all exact, i.e. they preserve short exact sequences in $\underline{s\mathcal{O}}$.

Lemma 3.6. *The supernatural transformation $x^* : sE \Rightarrow sE$ is both the left and right mate of $x : sF \Rightarrow sF$ with respect to the canonical adjunctions defined above.*

Proof. We just explain how to check that x^* is the left mate of x with respect to the adjunction (sE, sF); the argument for right mate is similar. We need to show for each $M \in \operatorname{ob} s\mathcal{O}$ that the composition

$$U^* \otimes M \xrightarrow{\mathrm{id} \otimes c_M} U^* \otimes U \otimes U^* \otimes M \xrightarrow{\mathrm{id} \otimes x_{U^* \otimes M}} U^* \otimes U \otimes U^* \otimes M \xrightarrow{d_{U^* \otimes M}} U^* \otimes M$$

is equal to $x^*_M : U^* \otimes M \to U^* \otimes M$. Recall for this that x^*_M is defined by left multiplication by $\sum_{r,s=1}^n \left(f'_{r,s} \otimes e'_{s,r} - f_{r,s} \otimes e_{s,r}\right)$, while $x_{U^* \otimes M}$ is defined by left multiplication by $\sum_{r,s=1}^n (f_{r,s} \otimes e_{s,r} \otimes 1 + f_{r,s} \otimes 1 \otimes e_{s,r} - f'_{r,s} \otimes e'_{s,r} \otimes 1 - f'_{r,s} \otimes 1 \otimes e'_{s,r})$. Now one computes the effect of both maps on homogeneous vectors of the form $\phi_t \otimes v$ and $\phi'_t \otimes v$ using (2.5)–(2.8). $\square$

4. Bruhat order

Consider the Dynkin diagram $\cdots\cdots \overset{-2}{\circ} \!-\!-\!-\! \overset{-1}{\circ} \!-\!-\!-\! \overset{0}{\circ} \!-\!-\!-\! \overset{1}{\circ} \!-\!-\!-\! \overset{2}{\circ} \cdots\cdots$ whose vertices are indexed by the totally ordered set $I = \mathbb{Z}$. We denote the associated Kac-Moody algebra by $\mathfrak{sl}_\infty$. This is the Lie algebra of traceless, finitely-supported complex matrices whose rows and columns are indexed by I. It is generated by the matrix units $f_i := e_{i+1,i}$ and $e_i := e_{i,i+1}$ for $i \in I$. The *natural representation* V^+ of $\mathfrak{sl}_\infty$ is the module of column vectors with standard basis $\{v_i^+ \mid i \in I\}$. We also need the *dual natural representation* V^- with basis $\{v_i^- \mid i \in I\}$. The action of the Chevalley generators on these bases is given by

$$e_i v_j^+ = \delta_{i+1,j} v_i^+, \qquad e_i v_j^- = \delta_{i,j} v_{i+1}^-, \tag{4.1}$$

$$f_i v_j^+ = \delta_{i,j} v_{i+1}^+, \qquad f_i v_j^- = \delta_{i+1,j} v_i^-. \tag{4.2}$$

The tensor product $V^{\otimes \boldsymbol{\sigma}} := V^{\sigma_1} \otimes \cdots \otimes V^{\sigma_n}$ has monomial basis $\{v_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}\}$ defined from $v_{\boldsymbol{b}} := v_{b_1}^{\sigma_1} \otimes \cdots \otimes v_{b_n}^{\sigma_n}$. Recalling (3.6), the Chevalley generators act on these monomials by

$$f_i v_{\boldsymbol{b}} = \sum_{\substack{1 \leq t \leq n \\ i\text{-}\mathrm{sig}_t(\boldsymbol{b}) = \mathtt{f}}} v_{\boldsymbol{b} + \sigma_t \boldsymbol{d}_t}, \qquad e_i v_{\boldsymbol{b}} = \sum_{\substack{1 \leq t \leq n \\ i\text{-}\mathrm{sig}_t(\boldsymbol{b}) = \mathtt{e}}} v_{\boldsymbol{b} - \sigma_t \boldsymbol{d}_t}. \tag{4.3}$$

This should be compared with Theorem 3.4, which already makes some connection between the endofunctors sF_j, sE_j of $s\mathcal{O}$ and the $\mathfrak{sl}_\infty$-module $V^{\otimes \boldsymbol{\sigma}}$.

We next introduce an important partial order $\succeq$ on $\mathbf{B}$, which we call the Bruhat order It is closely related to the inverse dominance order of [LW, Definition 3.2], which comes from Lusztig's construction of tensor products of based modules [L, §27.3]. The root system of $\mathfrak{sl}_\infty$ has *weight lattice* $P := \bigoplus_{i \in I} \mathbb{Z}\, \omega_i$ where ω_i is the ith fundamental weight. For $i \in I$, we set

$$\varepsilon_i := \omega_i - \omega_{i-1}, \qquad \alpha_i := \varepsilon_i - \varepsilon_{i+1}.$$

We identify ε_i with the weight of the vector v_i^+ in the $\mathfrak{sl}_\infty$-module V^+. Then, $v_i^- \in V^-$ is of weight $-\varepsilon_i$. For $\boldsymbol{b} \in \mathbf{B}$, let

$$\mathbf{wt}(\boldsymbol{b}) = (\mathrm{wt}_1(\boldsymbol{b}), \dots, \mathrm{wt}_n(\boldsymbol{b})) \in P^n \tag{4.4}$$

be the n-tuple of weights defined from $\mathrm{wt}_r(\boldsymbol{b}) := \sigma_r \varepsilon_{b_r}$, so that $v_{\boldsymbol{b}} \in V^{\otimes \boldsymbol{\sigma}}$ is of weight $|\,\mathbf{wt}(\boldsymbol{b})| := \mathrm{wt}_1(\boldsymbol{b}) + \cdots + \mathrm{wt}_n(\boldsymbol{b}) \in P$. Because the weight spaces of $V^\pm$ are all one-dimensional, the map $\mathbf{B} \to P^n, \boldsymbol{b} \mapsto \mathbf{wt}(\boldsymbol{b})$ is injective.

Definition 4.1. Let $\trianglelefteq$ denote the dominance order on P, so $\beta \trianglelefteq \gamma \Leftrightarrow \gamma - \beta \in \bigoplus_{i \in I} \mathbb{N}\,\alpha_i$. The *inverse dominance order* on P^n is the partial order defined by declaring that $(\beta_1, \dots, \beta_n) \succeq (\gamma_1, \dots, \gamma_n)$ if and only if

$$\beta_1 + \cdots + \beta_s \trianglelefteq \gamma_1 + \cdots + \gamma_s,$$

for each $s = 1, \dots, n$, with the inequality being an equality when $s = n$. Finally, define the *Bruhat order* $\succeq$ on $\mathbf{B}$ by $\boldsymbol{a} \succeq \boldsymbol{b} \Leftrightarrow \mathbf{wt}(\boldsymbol{a}) \succeq \mathbf{wt}(\boldsymbol{b})$.

Our first lemma makes the definition of the Bruhat order more explicit. Using it, one can check in particular that $\boldsymbol{a} \succeq \boldsymbol{b}$ implies that $\lambda_{\boldsymbol{a}} \geq \lambda_{\boldsymbol{b}}$ in the dominance order on $\mathfrak{t}^*$; cf. [BLW, Lemma 3.4].

Lemma 4.2. *For $\boldsymbol{a} \in \mathbf{B}, i \in I$ and $1 \leq s \leq n$, we let*

$$N_{[1,s]}(\boldsymbol{a}, i) := \#\{1 \leq r \leq s \mid a_r > i, \sigma_r = +\} - \#\{1 \leq r \leq s \mid a_r > i, \sigma_r = -\}.$$

Then we have that $\boldsymbol{a} \succeq \boldsymbol{b}$ if and only if

- $N_{[1,n]}(\boldsymbol{a}, i) = N_{[1,n]}(\boldsymbol{b}, i)$ *for all $i \in I$;*
- $N_{[1,s]}(\boldsymbol{a}, i) \geq N_{[1,s]}(\boldsymbol{b}, i)$ *for all $i \in I$ and $s = 1, \dots, n-1$.*

Proof. This is a special case of [BLW, Lemma 2.17]. □

Lemma 4.3. *Assume that $\boldsymbol{a} \succeq \boldsymbol{b}$ and i-$\mathrm{sig}_r(\boldsymbol{a}) = i$-$\mathrm{sig}_n(\boldsymbol{b}) = \mathtt{f}$ for some $i \in I$ and $1 \leq r \leq n$. Then $\boldsymbol{a} + \sigma_r \boldsymbol{d}_r \succeq \boldsymbol{b} + \sigma_n \boldsymbol{d}_n$, with equality if and only if $\boldsymbol{a} = \boldsymbol{b}$ and $r = n$.*

Proof. We use the conditions from Lemma 4.2. For *either* $j \neq i$ and $1 \leq s \leq n$, *or* $j = i$ and $1 \leq s < r$, we have that

$$N_{[1,s]}(\boldsymbol{a} + \sigma_r \boldsymbol{d}_r, j) = N_{[1,s]}(\boldsymbol{a}, j) \geq N_{[1,s]}(\boldsymbol{b}, j) = N_{[1,s]}(\boldsymbol{b} + \sigma_n \boldsymbol{d}_n, j).$$

For $r \leq s < n$, we have that

$$N_{[1,s]}(\boldsymbol{a} + \sigma_r \boldsymbol{d}_r, i) = N_{[1,s]}(\boldsymbol{a}, i) + 1 \geq N_{[1,s]}(\boldsymbol{b}, i) + 1 > N_{[1,s]}(\boldsymbol{b}, i) = N_{[1,s]}(\boldsymbol{b} + \sigma_n \boldsymbol{d}_n, i).$$

Finally, $N_{[1,n]}(\boldsymbol{a} + \sigma_r \boldsymbol{d}_r, i) = N_{[1,n]}(\boldsymbol{a}, i) + 1 = N_{[1,n]}(\boldsymbol{b}, i) + 1 = N_{[1,n]}(\boldsymbol{b} + \sigma_n \boldsymbol{d}_n, i)$. □

To prepare for the next lemma, suppose that we are given $\boldsymbol{b} \in \mathbf{B}$. Define $\boldsymbol{a} \in \mathbf{B}$ by setting $a_1 := b_1$, then inductively defining a_s for $s = 2, \dots, n$ as follows.

- If $\sigma_s = +$ then a_s is the greatest integer such that $a_s \leq b_s$, and the following hold for all $1 \leq r < s$:
  - if $\sigma_r = +$ then $a_s < a_r$;
  - if $\sigma_r = -$ then $a_s < b_r$.
- If $\sigma_s = -$ then a_s is the smallest integer such that $a_s \geq b_s$, and the following hold for all $1 \leq r < s$:
  - if $\sigma_r = -$ then $a_s > a_r$;
  - if $\sigma_r = +$ then $a_s > b_r$.

Also define a monomial $X = X_n \cdots X_2$ in the Chevalley generators $\{f_i \,|\, i \in I\}$ by setting

$$X_r := \begin{cases} f_{b_r-1} \cdots f_{a_r+1} f_{a_r} & \text{if } \sigma_r = +, \\ f_{b_r} \cdots f_{a_r-2} f_{a_r-1} & \text{if } \sigma_r = -, \end{cases}$$

for each $r = 2, \dots, n$.

Example 4.4. If $\boldsymbol{\sigma} = (+,+,-,+,-,-)$ and $\boldsymbol{b} = (3,4,3,4,3,4)$, then $\boldsymbol{a} = (3,2,5,1,6,7)$ and $X = (f_4f_5f_6)(f_3f_4f_5)(f_3f_2f_1)(f_3f_4)(f_3f_2)$.

Lemma 4.5. *In the above notation, we have that $Xv_{\boldsymbol{a}} = v_{\boldsymbol{b}} + ($a sum of $v_{\boldsymbol{c}}$'s for $\boldsymbol{c} \succ \boldsymbol{b})$.*

Proof. We proceed by induction on n, the result being trivial in case $n = 1$. For $n > 1$, let $\bar{\boldsymbol{\sigma}} := (\sigma_1, \dots, \sigma_{n-1})$, $\bar{\boldsymbol{a}} := (a_1, \dots, a_{n-1})$, $\bar{\boldsymbol{b}} := (b_1, \dots, b_{n-1})$ and $\bar{X} := X_{n-1} \cdots X_2$. Applying the induction hypothesis in the $\mathfrak{sl}_\infty$-module $V^{\otimes \bar{\boldsymbol{\sigma}}}$, we get that

$$\bar{X} v_{\bar{\boldsymbol{a}}} = v_{\bar{\boldsymbol{b}}} + (\text{a sum of } v_{\bar{\boldsymbol{c}}}\text{'s for } \bar{\boldsymbol{c}} \succ \bar{\boldsymbol{b}}).$$

Now we observe that if f_i is a Chevalley generator appearing in one of the monomials X_r for $r < n$, then $f_i v_{a_n}^{\sigma_n} = 0$. This follows from the definitions: if $\sigma_n = +$ we must show that $i \neq a_n$, which follows as $i \geq a_r > a_n$ if $\sigma_r = +$ or $i \geq b_r > a_n$ if $\sigma_r = -$; if $\sigma_n = -$ we must show that $i \neq a_n - 1$, which follows as $i < b_r < a_n$ if $\sigma_r = +$ or $i < a_r < a_n$ if $\sigma_r = -$. Hence, letting $\tilde{\boldsymbol{b}} := (b_1, \dots, b_{n-1}, a_n)$, we deduce that

$$\bar{X} v_{\boldsymbol{a}} = v_{\tilde{\boldsymbol{b}}} + (\text{a sum of } v_{\boldsymbol{c}}\text{'s for } \boldsymbol{c} \succ \tilde{\boldsymbol{b}}).$$

Finally we act with X_n, which sends $v_{a_n}^{\sigma_n}$ to $v_{b_n}^{\sigma_n}$, and apply Lemma 4.3. □

Theorem 4.6. *For every $\boldsymbol{b} \in \mathbf{B}$, the indecomposable projective supermodule $P(\boldsymbol{b})$ has a Verma flag with top section evenly isomorphic to $M(\boldsymbol{b})$ and other sections evenly isomorphic to $M(\boldsymbol{c})$'s for $\boldsymbol{c} \in \mathbf{B}$ with $\boldsymbol{c} \succ \boldsymbol{b}$.*

Proof. Let notation be as in Lemma 4.5. Let $i_1, \dots, i_l \in I$ be defined so that X is the monomial $f_{i_l} \cdots f_{i_2} f_{i_1}$. Let $j_k := \sqrt{z + i_k}\sqrt{z + i_k + 1}$ for each k and consider the supermodule

$$P := sF_{j_l} \cdots sF_{j_2} sF_{j_1} M(\boldsymbol{a}).$$

For each $1 \leq r < s \leq n$, we have that $a_r > a_s$ if $\sigma_r = \sigma_s = +$, $a_r < a_s$ if $\sigma_r = \sigma_s = -$, and $a_r \neq a_s$ if $\sigma_r \neq \sigma_s$. This implies that the weight $\lambda_{\boldsymbol{a}}$ is typical and dominant, hence $M(\boldsymbol{a})$ is projective by Lemma 2.4. Since each sF_j sends projectives to projectives, we deduce that P is projective. Since the combinatorics of (4.3) matches that of Theorem 3.4, we can reinterpret Lemma 4.5 as saying that P has a Verma flag with one section evenly isomorphic to $M(\boldsymbol{b})$ and all other sections evenly isomorphic to $M(\boldsymbol{c})$'s for $\boldsymbol{c} \succ \boldsymbol{b}$. In fact, the unique section isomorphic to $M(\boldsymbol{b})$ appears at the top of this Verma flag, thanks the order of the sections arising from Theorem 3.4(1). Hence, P has a summand evenly isomorphic to $P(\boldsymbol{b})$, and it just remains to apply Lemma 2.5. □

Corollary 4.7. *For $\boldsymbol{b} \in \mathbf{B}$, we have that $[M(\boldsymbol{b}) : L(\boldsymbol{b})] = 1$. All other composition factors of $M(\boldsymbol{b})$ are evenly isomorphic to $L(\boldsymbol{c})$'s for $\boldsymbol{c} \prec \boldsymbol{b}$.*

Proof. This follows from Theorem 4.6 and the following analog of *BGG reciprocity*: for $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}$, we have that

$$[M(\boldsymbol{b}) : L(\boldsymbol{a})] = \dim \mathrm{Hom}_{s\mathcal{O}}(P(\boldsymbol{a}), M(\boldsymbol{b})^\star)_{\bar{0}} = (P(\boldsymbol{a}) : M(\boldsymbol{b})),$$
$$[M(\boldsymbol{b}) : \Pi L(\boldsymbol{a})] = \dim \mathrm{Hom}_{s\mathcal{O}}(P(\boldsymbol{a}), M(\boldsymbol{b})^\star)_{\bar{1}} = (P(\boldsymbol{a}) : \Pi M(\boldsymbol{b})).$$

The various equalities here follow from Lemmas 2.5 and 2.3. □

Corollary 4.8. *For any $\boldsymbol{b} \in \mathbf{B}$, every irreducible subquotient of the indecomposable projective $P(\boldsymbol{b})$ is evenly isomorphic to $L(\boldsymbol{a})$ for $\boldsymbol{a} \in \mathbf{B}$ with $|\,\mathbf{wt}(\boldsymbol{a})| = |\,\mathbf{wt}(\boldsymbol{b})|$.*

Proof. By Theorem 4.6, $P(\boldsymbol{b})$ has a Verma flag with sections $M(\boldsymbol{c})$ for $\boldsymbol{c} \succeq \boldsymbol{b}$. By Corollary 4.7, the composition factors of $M(\boldsymbol{c})$ are $L(\boldsymbol{a})$'s for $\boldsymbol{a} \preceq \boldsymbol{c}$. Hence, every irreducible subquotient of $P(\boldsymbol{b})$ is evenly isomorphic to $L(\boldsymbol{a})$ for $\boldsymbol{a} \in \mathbf{B}$ such that $\boldsymbol{a} \preceq \boldsymbol{c} \succeq \boldsymbol{b}$ for some $\boldsymbol{c}$. This condition implies that $|\,\mathbf{wt}(\boldsymbol{a})| = |\,\mathbf{wt}(\boldsymbol{b})|$. □

5. Weak categorical action

Let $\mathcal{O}$ be the Serre subcategory of $s\mathcal{O}$ generated by $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$, i.e. it is the full subcategory of $s\mathcal{O}$ consisting of all supermodules whose composition factors are evenly isomorphic to $L(\boldsymbol{b})$'s for $\boldsymbol{b} \in \mathbf{B}$. Since each $L(\boldsymbol{b})$ is of type $\mathtt{M}$, there are no non-zero odd morphisms between objects of $\mathcal{O}$. Because of this, we forget the $\mathbb{Z}/2$-grading and simply view $\mathcal{O}$ as a $\Bbbk$-linear category rather than a supercategory.

Theorem 5.1. *We have that $s\mathcal{O} = \mathcal{O} \oplus \Pi\mathcal{O}$ in the sense defined in the introduction.*

Proof. Let $\Pi\mathcal{O}$ be the Serre subcategory of $s\mathcal{O}$ generated by $\{\Pi L(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathbf{B}\}$. By Corollary 4.8, all even extensions between $\Pi L(\boldsymbol{a})$ and $L(\boldsymbol{b})$ are split. Hence, every supermodule in $s\mathcal{O}$ decomposes uniquely as a direct sum of an object of $\mathcal{O}$ and an object of $\Pi\mathcal{O}$. The result follows. □

Remark 5.2. For typical blocks, Theorem 5.1 has a more direct proof exploiting the action of the anticenter of $U(\mathfrak{g})$; see [F, §3.1].

In order to state our next theorem, we briefly recall the following definition due to Cline, Parshall and Scott [CPS]:

Definition 5.3. A *highest weight category* is a Schurian category $\mathcal{C}$ in the sense of Definition 2.2, together with an interval-finite poset $(\Lambda, \leq)$ indexing a complete set of irreducible objects $\{L(\lambda) \mid \lambda \in \Lambda\}$, subject to the following axiom. For each $\lambda \in \Lambda$, let $P(\lambda)$ be a projective cover of $L(\lambda)$ in $\mathcal{C}$. Define the *standard object* $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\lambda)] = 1$ and $[\Delta(\lambda) : L(\mu)] = 0$ for $\mu \not\leq \lambda$. Then we require that $P(\lambda)$ has a filtration with top section isomorphic to $\Delta(\lambda)$ and all other sections of the form $\Delta(\mu)$ for $\mu > \lambda$.

Theorem 5.4. *The category $\mathcal{O}$ is a highest weight category with weight poset $(\mathbf{B}, \preceq)$. Its standard objects are the Verma supermodules $\{M(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$.*

Proof. It is clear that $\mathcal{O}$ is a Schurian category with isomorphism classes of irreducible objects represented by $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$. By Theorem 4.6, $P(\boldsymbol{b})$ has a Verma flag with $M(\boldsymbol{b})$ at the top and other sections that are evenly isomorphic to $M(\boldsymbol{c})$'s for $\boldsymbol{c} \succ \boldsymbol{b}$. It just remains to observe that the Verma supermodules $M(\boldsymbol{b})$ coincide with the standard objects $\Delta(\boldsymbol{b})$. This follows using the filtration just described plus Corollary 4.7. □

Remark 5.5. By Lemma 2.3, the duality $\star$ on $s\mathcal{O}$ restricts to a duality $\star : \mathcal{O} \to \mathcal{O}$ fixing isomorphism classes of irreducible objects.

Next, take $i \in I$ and set $j := \sqrt{z+i}\sqrt{z+i+1}$. Theorem 3.4 implies that the exact functors sF_j and sE_j send the standard objects in $\mathcal{O}$ to objects of $\mathcal{O}$ with a Verma flag.

Hence, they send arbitrary objects in $\mathcal{O}$ to objects of $\mathcal{O}$. Thus, their restrictions define endofunctors

$$F_i := sF_j|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}, \quad E_i := sE_j|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}. \tag{5.1}$$

Again, these functors are both left and right adjoint to each other. Let $\mathcal{O}^\Delta$ be the full subcategory of $\mathcal{O}$ consisting of all objects possessing a Verma flag. This is an exact subcategory of $\mathcal{O}$. Its complexified Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{O}^\Delta)$ has basis $\{[M(\boldsymbol{b})] \mid \boldsymbol{b} \in \mathbf{B}\}$.

Theorem 5.6. *For each $i \in I$, the functors F_i and E_i are exact endofunctors of $\mathcal{O}^\Delta$. Moreover, if we identify $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{O}^\Delta)$ with $V^{\otimes \boldsymbol{\sigma}}$ so $[M(\boldsymbol{b})] \leftrightarrow v_{\boldsymbol{b}}$ for each $\boldsymbol{b} \in \mathbf{B}$, then the induced endomorphisms $[F_i]$ and $[E_i]$ of the Grothendieck group act in the same way as the Chevalley generators f_i and e_i of $\mathfrak{sl}_\infty$.*

Proof. Compare Theorem 3.4 with (4.3). $\square$

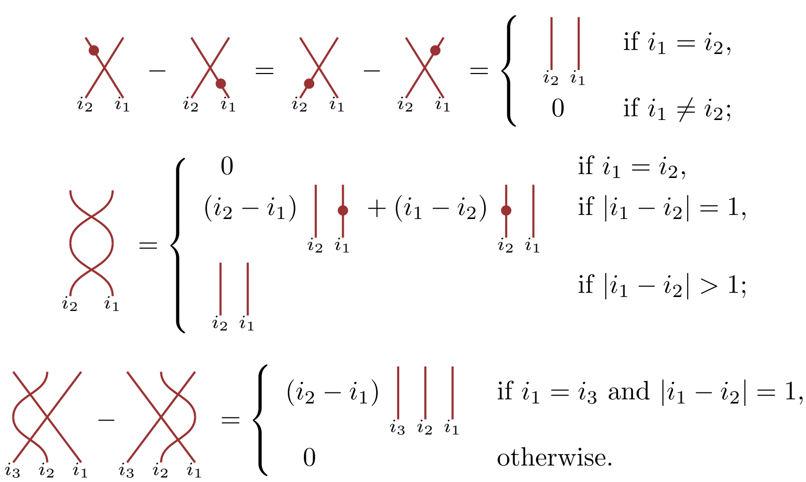
Thus, we have constructed a highest weight category $\mathcal{O}$ with weight poset $(\mathbf{B}, \preceq)$, and equipped it with a weak categorical action of the Lie algebra $\mathfrak{sl}_\infty$ in the sense of [CR, R].

6. Strong categorical action

In this section, we upgrade the weak categorical action of $\mathfrak{sl}_\infty$ on $\mathcal{O}$ constructed so far to a strong categorical action. For the following definition, we represent morphisms in a strict monoidal category via the usual string calculus, adopting the same conventions for horizontal and vertical composition as [KL].

Definition 6.1. The *quiver Hecke category* of type $\mathfrak{sl}_\infty$ is the strict $\Bbbk$-linear monoidal category $\mathcal{QH}$ with objects generated by the set I, and morphisms generated by  $: i \to i$

and  $: i_2 \otimes i_1 \to i_1 \otimes i_2$, subject to the following relations:



Let I^d denote the set of words $\boldsymbol{i} = i_d \cdots i_1$ of length d in the alphabet I, and identify $\boldsymbol{i} \in I^d$ with the object $i_d \otimes \cdots \otimes i_1 \in \operatorname{ob} \mathcal{QH}$. Then, the locally unital algebra

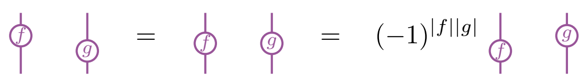
$$QH_d := \bigoplus_{\boldsymbol{i}, \boldsymbol{i}' \in I^d} \operatorname{Hom}_{\mathcal{QH}}(\boldsymbol{i}, \boldsymbol{i}') \tag{6.1}$$

is the *quiver Hecke algebra* of type $\mathfrak{sl}_\infty$ defined originally by Khovanov and Lauda [KL] and Rouquier [R].

Recall for a $\Bbbk$-linear category $\mathcal{C}$ that there is an associated strict $\Bbbk$-linear monoidal category $\mathcal{E}nd(\mathcal{C})$ consisting of $\Bbbk$-linear endofunctors and natural transformations. The remainder of the section will be devoted to the proof of the following theorem.

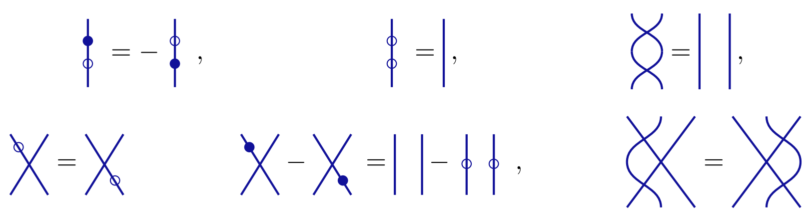
Theorem 6.2. *There is a strict monoidal functor $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$ sending the generating objects $i \in I$ to the endofunctors F_i from (5.1). Moreover, for all $M \in \operatorname{ob}\mathcal{O}$ and $i \in I$, the endomorphism $F_iM \to F_iM$ defined by the natural transformation* $\Phi\Big($  $\Big)$ *is nilpotent.*

In order to construct Φ, we need to pass through two intermediate objects $\mathcal{AHC}$, the (degenerate) affine Hecke-Clifford supercategory, and $\mathcal{QHC}$, which is a certain quiver Hecke-Clifford supercategory in the sense of [KKT]. Both $\mathcal{AHC}$ and $\mathcal{QHC}$ are examples of (strict) *monoidal supercategories*, meaning that they are supercategories equipped with a monoidal product in an appropriate enriched sense. We refer the reader to the introduction of [BE1] for the precise definition, just recalling that morphisms in a monoidal supercategory satisfy the *super interchange law* rather than the usual interchange law of a monoidal category: in terms of the string calculus as in [BE1] we have that

 (6.2)

for homogeneous morphisms f and g of parities $|f|$ and $|g|$, respectively.

Definition 6.3. The (degenerate) *affine Hecke-Clifford supercategory* $\mathcal{AHC}$ is the strict monoidal supercategory with a single generating object 1, even generating morphisms  $: 1 \to 1$ and  $: 1 \otimes 1 \to 1 \otimes 1$, and an odd generating morphism  $: 1 \to 1$. These are subject to the following relations:

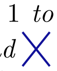


Denoting the object $1^{\otimes d} \in \operatorname{ob}\mathcal{AHC}$ simply by d, the (degenerate) *affine Hecke-Clifford superalgebra* is the superalgebra

$$AHC_d := \operatorname{End}_{\mathcal{AHC}}(d). \tag{6.3}$$

This was introduced originally by Nazarov [N, §3].

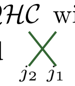
For a supercategory $\mathcal{C}$, we write $\mathcal{E}nd(\mathcal{C})$ for the strict monoidal supercategory consisting of superfunctors and supernatural transformations.

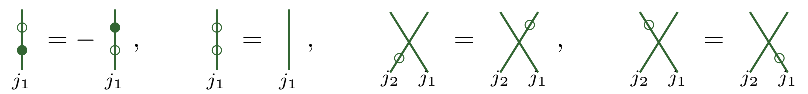
Theorem 6.4. *There is a strict monoidal superfunctor $\Psi : \mathcal{AHC} \to \mathcal{E}nd(s\mathcal{O})$ sending the generating object 1 to the endofunctor $sF = U \otimes -$ from (3.1), and the generating morphisms*  *,*  *and*  *to the supernatural transformations x, c and t which are defined on $M \in \operatorname{ob} s\mathcal{O}$ as follows:*

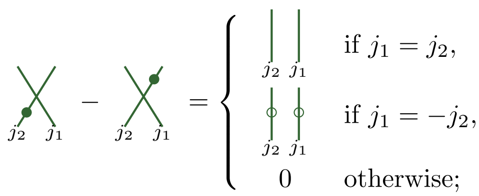
- *$x_M : U \otimes M \to U \otimes M$ is left multiplication by the tensor ω from (3.2);*

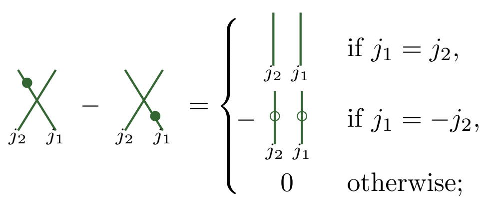
- *$c_M : U \otimes M \to U \otimes M$ is left multiplication by $\sqrt{-1}\, f' \otimes 1$ for f' as in (3.4);*
- *$t_M : U \otimes U \otimes M \to U \otimes U \otimes M$ sends $u \otimes v \otimes m \mapsto (-1)^{|u||v|} v \otimes u \otimes m$.*

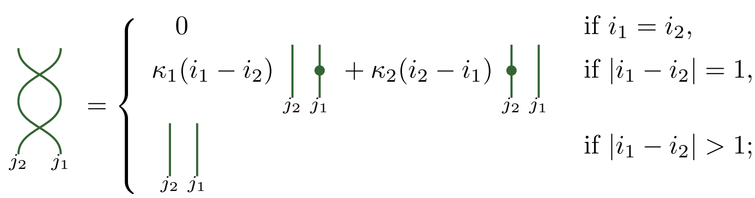
Proof. This an elementary check of relations, similar to the one made in the proof of [HKS, Theorem 7.4.1]. □

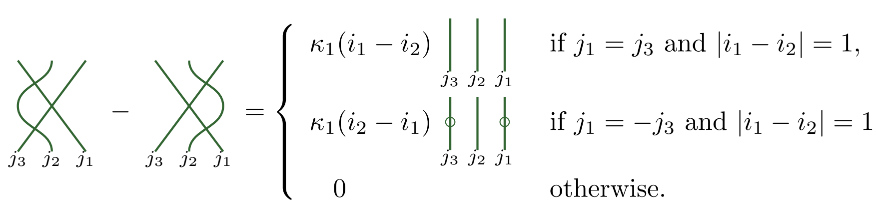
Definition 6.5. The *quiver Hecke-Clifford supercategory* of type $\mathfrak{sl}_\infty$ is the monoidal supercategory $\mathcal{QHC}$ with objects generated by the set J, even generating morphisms  $: j_1 \to j_1$ and  $: j_2 \otimes j_1 \to j_1 \otimes j_2$, and odd generating morphisms  $: j_1 \to -j_1$, for all $j_1, j_2 \in J$. These are subject to the following relations:











In the above, we have adopted the convention given $j_r \in J$ that $i_r \in I$ and $\kappa_r \in \{\pm 1\}$ are defined from $j_r = \kappa_r \sqrt{z + i_r}\sqrt{z + i_r + 1}$. Identifying the word $\boldsymbol{j} = j_d \cdots j_1 \in J^d$ with $j_d \otimes \cdots \otimes j_1 \in \operatorname{ob} \mathcal{QHC}$, the *quiver Hecke-Clifford superalgebra* is the locally unital algebra

$$QHC_d := \bigoplus_{\boldsymbol{j}, \boldsymbol{j}' \in J^d} \operatorname{Hom}_{\mathcal{QHC}}(\boldsymbol{j}, \boldsymbol{j}'). \tag{6.4}$$

This is exactly as in [KKT, Definition 3.5] in the special case of the $\mathfrak{sl}_\infty$-quiver.

Now we are going to exploit a remarkable isomorphism between certain completions $\widehat{AHC}_d$ and $\widehat{QHC}_d$ of the superalgebras AHC_d and QHC_d from (6.3) and (6.4), which was constructed in [KKT]. To define these, we need some further notation.

Numbering strands of a diagram by $1, \dots, d$ from right to left, AHC_d is generated by its elements x_r, c_r $(1 \leq r \leq d)$ and t_r $(1 \leq r < d)$ corresponding to the closed dot on the rth strand, the open dot on the rth strand, and the crossing of the rth and $(r+1)$th strands, respectively. Let $HC_d := S_d \ltimes C_d$ be the *Sergeev superalgebra*, that is, the smash product of the symmetric group S_d with basic transpositions $t_1, \dots, t_{d-1}$ acting on the Clifford superalgebra C_d on generators $c_1, \dots, c_d$. Let A_d denote the purely even polynomial superalgebra $\Bbbk[x_1, \dots, x_d]$. By the basis theorem for AHC_d established in [BK, §2-k], the natural multiplication map gives a superspace isomorphism $HC_d \otimes A_d \stackrel{\sim}{\to} AHC_d$. Transporting the multiplication on AHC_d to $HC_d \otimes A_d$ via this isomorphism, the following describe how to commute a polynomial $f \in A_d$ past the generators of HC_d:

$$(1 \otimes f)(c_r \otimes 1) = c_r \otimes c_r(f), \tag{6.5}$$

$$(1 \otimes f)(t_r \otimes 1) = t_r \otimes t_r(f) + 1 \otimes \partial_r(f) + c_r c_{r+1} \otimes \tilde{\partial}_r(f), \tag{6.6}$$

for operators $c_r, t_r, \partial_r, \tilde{\partial}_r : A_d \to A_d$ such that

- t_r is the automorphism that interchanges x_r and x_{r+1} and fixes all other generators;
- c_r is the automorphism that sends $x_r \mapsto -x_r$ and fixes all other generators;
- ∂_r is the Demazure operator $\partial_r(f) := \frac{t_r(f) - f}{x_r - x_{r+1}}$;
- $\tilde{\partial}_r$ is the twisted Demazure operator $c_{r+1} \circ \partial_r \circ c_r$, so $\tilde{\partial}_r(f) = \frac{t_r(f) - c_{r+1}(c_r(f))}{x_r + x_{r+1}}$.

Given a tuple $\mu = (\mu_i)_{i \in I}$ of non-negative integers all but finitely many of which are zero, the quotient superalgebra

$$AHC_d(\mu) := AHC_d \Big/ \Big\langle \prod_{i \in I} \left(x_1^2 - (z+i)(z+i+1)\right)^{\mu_i} \Big\rangle \tag{6.7}$$

is a (degenerate) *cyclotomic Hecke-Clifford superalgebra* in the sense of [BK, §3.e]. It is finite dimensional. Moreover, all roots of the minimal polynomials of all $x_r \in AHC_d(\mu)$ belong to the set J. It follows for each $\boldsymbol{j} = j_d \cdots j_1$ in the set J^d of words of length d in letters J that there is an idempotent $1_{\boldsymbol{j}} \in AHC_d(\mu)$ defined by the projection onto the simultaneous generalized eigenspaces for $x_1, \dots, x_d$ with eigenvalues $j_1, \dots, j_d$, respectively. Moreover, we have that

$$AHC_d(\mu) = \bigoplus_{\boldsymbol{j}, \boldsymbol{j}' \in J^d} 1_{\boldsymbol{j}'} AHC_d(\mu) 1_{\boldsymbol{j}}.$$

If $\mu \leq \mu'$, i.e. $\mu_i \leq \mu_i'$ for all i, there is a canonical surjection $AHC_d(\mu') \twoheadrightarrow AHC_d(\mu)$ sending $x_r, c_r, t_r, 1_{\boldsymbol{j}} \in AHC_d(\mu')$ to the elements of $AHC_d(\mu)$ with the same names. Let

$$\widehat{AHC}_d := \varprojlim_{\mu} AHC_d(\mu) \tag{6.8}$$

be the inverse limit of this system of superalgebras taken in the category of locally unital superalgebras with distinguished idempotents indexed by J^d. Using the basis theorem for the cyclotomic quotients $AHC_d(\mu)$ from [BK, §3-e], one can identify $\widehat{AHC}_d$ with the completion defined in [KKT, Definition 5.3][1]. In particular, letting

$$\widehat{A}_d := \bigoplus_{\boldsymbol{j}\in J^d} \Bbbk[[x_1 - j_1, \dots, x_d - j_d]]1_{\boldsymbol{j}},$$

there is a superspace isomorphism $HC_d \otimes \widehat{A}_d \xrightarrow{\sim} \widehat{AHC}_d$ induced by the obvious multiplication maps $HC_d \otimes \widehat{A}_d \twoheadrightarrow AHC_d(\mu)$ for all μ. The multiplication on $HC_d \otimes \widehat{A}_d$ corresponding to the one on $\widehat{AHC}_d$ via this isomorphism has the following properties for all $f \in \widehat{A}_d$:

$$(1 \otimes f1_{\boldsymbol{j}})(c_r \otimes 1_{\boldsymbol{j}'}) = c_r \otimes c_r(f)1_{c_r(\boldsymbol{j})}1_{\boldsymbol{j}'}, \tag{6.9}$$

$$\begin{aligned}(1 \otimes f1_{\boldsymbol{j}})(t_r \otimes 1_{\boldsymbol{j}'}) = t_r \otimes t_r(f)1_{t_r(\boldsymbol{j})}1'_{\boldsymbol{j}} + 1 \otimes \frac{t_r(f)1_{t_r(\boldsymbol{j})} - f1_{\boldsymbol{j}}}{x_r - x_{r+1}}1_{\boldsymbol{j}'}\\ + c_rc_{r+1} \otimes \frac{t_r(f)1_{t_r(\boldsymbol{j})} - c_{r+1}(c_r(f))1_{c_{r+1}(c_r(\boldsymbol{j}))}}{x_r + x_{r+1}}1_{\boldsymbol{j}'}.\end{aligned} \tag{6.10}$$

The fractions on the right hand side of (6.10) make sense: in the first, $(x_r - x_{r+1})1_{\boldsymbol{j}'}$ is invertible unless $j'_r = j'_{r+1}$, in which case the expression equals $\partial_r(f)1_{\boldsymbol{j}}1_{\boldsymbol{j}'}$; the second is fine when $j'_r \neq -j'_{r+1}$ as then $(x_r + x_{r+1})1_{\boldsymbol{j}'}$ is invertible, while if $j'_r = -j'_{r+1}$ it equals $\tilde{\partial}_r(f)1_{t_r(\boldsymbol{j})}1_{\boldsymbol{j}'}$.

Similarly, there is a completion $\widehat{QHC}_d$ of QHC_d. To introduce this, we denote the elements of $QHC_d1_{\boldsymbol{j}}$ defined by an open dot on the rth strand, a closed dot on the rth strand and a crossing of the rth and $(r+1)$th strands by $\gamma_r1_{\boldsymbol{j}}, \xi_r1_{\boldsymbol{j}}$ and $\tau_r1_{\boldsymbol{j}}$, respectively. For $\mu = (\mu_i)_{i\in I}$ as above, we define the *cyclotomic quiver Hecke-Clifford superalgebra*

$$QHC_d(\mu) := QHC_d\Big/\left\langle \xi_1^{2\mu_i}1_{\boldsymbol{j}} \;\middle|\; \boldsymbol{j} \in J^d, i \in I \text{ with } j_1^2 = (z+i)(z+i+1)\right\rangle. \tag{6.11}$$

Using the relations, it is easy to see that the images of all $\xi_r1_{\boldsymbol{j}}$ are nilpotent in $QHC_d(\mu)$. Then we set

$$\widehat{QHC}_d := \varprojlim_{\mu} QHC_d(\mu), \tag{6.12}$$

taking the inverse limit once again in the category of locally unital superalgebras with distinguished idempotents indexed by J^d. The obvious locally unital homomorphisms $QHC_d\otimes_{\Bbbk[\xi_1,\dots,\xi_d]}\Bbbk[[\xi_1,\dots,\xi_d]] \twoheadrightarrow QHC_d(\mu)$ for each μ induce a surjective homomorphism

$$QHC_d \otimes_{\Bbbk[\xi_1,\dots,\xi_d]} \Bbbk[[\xi_1,\dots,\xi_d]] \to \widehat{QHC}_d.$$

This map is actually an isomorphism, as may be deduced using the basis theorem for QHC_d from [KKT, Corollary 3.9] plus the observation that the image of any non-zero element $u \in QHC_d$ is non-zero in $QHC_d(\mu)$ for sufficiently large μ; the latter assertion follows by elementary considerations involving the natural $\mathbb{Z}$-grading on QHC_d. Consequently, $\widehat{QHC}_d$ is isomorphic to the completion introduced in a slightly different way in [KKT, Definition 3.16]. Moreover, there is a locally unital embedding $QHC_d \hookrightarrow \widehat{QHC}_d$.

At last, we are ready to state the crucial theorem from [KKT]. We need this only in the special situation of [KKT, §5.2(i)(a)], but emphasize that the results obtained in [KKT] are substantially more general. In particular, for us, all elements of the set I are

[1] Note there is a sign error in [KKT, (5.5)]: it should read $-C_aC_{a+1}\dots$.

even in the sense of [KKT, §3.5], so that we do not need the more general quiver Hecke *super*algebras of [KKT].

Theorem 6.6. *There is a superalgebra isomorphism* $\widehat{QHC}_d \xrightarrow{\sim} \widehat{AHC}_d$ *such that*

$$1_{\boldsymbol{j}} \mapsto 1_{\boldsymbol{j}}, \quad \gamma_r 1_{\boldsymbol{j}} \mapsto c_r 1_{\boldsymbol{j}}, \quad \xi_r 1_{\boldsymbol{j}} \mapsto y_r 1_{\boldsymbol{j}}, \quad \tau_r 1_{\boldsymbol{j}} \mapsto t_r g_r 1_{\boldsymbol{j}} + f_r 1_{\boldsymbol{j}} + c_r c_{r+1} \tilde{f}_r 1_{\boldsymbol{j}},$$

for all $\boldsymbol{j} \in J^d$ *and* r. *Here,* $y_r \in \Bbbk[[x_r - j_r]]$ *and* $g_r, f_r, \tilde{f}_r \in \Bbbk[[x_r - j_r, x_{r+1} - j_{r+1}]]$ *are the power series determined uniquely by the following:*

$$\begin{aligned}
&j_r = \kappa_r \sqrt{z+i_r}\sqrt{z+i_r+1} \text{ for } i_r \in I \text{ and } \kappa_r \in \{\pm\},\\
&y_r = \kappa_r \left(\sqrt{x_r^2 + \tfrac{1}{4}} - \left(z + i_r + \tfrac{1}{2}\right)\right) \in (x_r - j_r),\\
&p_r = \frac{(x_r^2 - x_{r+1}^2)^2}{2(x_r^2 + x_{r+1}^2) - (x_r^2 - x_{r+1}^2)^2},\\
&g_r = \begin{cases} -1 & \text{if } i_r < i_{r+1},\\ p_r\left(\kappa_r y_r - \kappa_{r+1} y_{r+1}\right) & \text{if } i_r = i_{r+1}+1,\\ p_r & \text{if } i_r > i_{r+1}+1,\\ \frac{\sqrt{p_r}}{y_r - y_{r+1}} \in \frac{x_r - x_{r+1}}{y_r - y_{r+1}} + (x_r - x_{r+1}) & \text{if } j_r = j_{r+1},\\ \frac{\sqrt{p_r}}{y_r + y_{r+1}} \in \frac{x_r + x_{r+1}}{y_r + y_{r+1}} + (x_r + x_{r+1}) & \text{if } j_r = -j_{r+1}; \end{cases}\\
&f_r = \frac{g_r}{x_r - x_{r+1}} - \frac{\delta_{j_r, j_{r+1}}}{y_r - y_{r+1}}, \qquad \tilde{f}_r = \frac{g_r}{x_r + x_{r+1}} - \frac{\delta_{j_r, -j_{r+1}}}{y_r + y_{r+1}}.
\end{aligned}$$

(All of this notation depends implicitly on $\boldsymbol{j}$*.)*

Proof. This is a special case of [KKT, Theorem 5.4]. To help the reader to translate between our notation and that of [KKT], we note that the set J in [KKT] is the same as our set J, but the set I there is $\tilde{I} := \{j^2 \mid j \in J\}$, which is different from our I. We have made various other choices as stipulated in [KKT] in order to produce concrete formulae: we have taken the functions $\varepsilon : J \to \{0,1\}$ and $h : \tilde{I} \to \Bbbk$ from [KKT, (5.7)] so that $\varepsilon(j) = (1-\kappa)/2$ and $h(j^2) = z + i + \frac{1}{2}$ for $J \ni j = \kappa\sqrt{z+i}\sqrt{z+i+1}$; for [KKT, (5.11)] we took $G_{j_r, j_{r+1}}$ (our g_r) to be -1 when $i_r < i_{r+1}$. The fact that g_r, f_r and $\tilde{f}_r$ are all well-defined elements of $\Bbbk[[x_r - j_r, x_{r+1} - j_{r+1}]]$ is justified by [KKT, Lemma 5.5]. Note also that the ambiguous square roots appearing in the formulae for y_r and g_r are uniquely determined by the containments we have specified. □

Proof of Theorem 6.2. For $\boldsymbol{i} = i_d \cdots i_1 \in I^d$, let $F_{\boldsymbol{i}} := F_{i_d} \cdots F_{i_1} : \mathcal{O} \to \mathcal{O}$. The usual vertical composition of natural transformations makes the vector space

$$NT_d := \bigoplus_{\boldsymbol{i}, \boldsymbol{i}' \in I^d} \operatorname{Hom}(F_{\boldsymbol{i}}, F_{\boldsymbol{i}'})$$

into a locally unital algebra with distinguished idempotents $\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^d\}$ arising from the identity endomorphisms of each $F_{\boldsymbol{i}}$. Also horizontal composition of natural transformations defines homomorphisms $a_{d_2,d_1} : NT_{d_2} \otimes NT_{d_1} \to NT_{d_2+d_1}$ for all $d_1, d_2 \geq 0$. Recalling (6.1), the data of a strict monoidal functor $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$ sending i to F_i is just the same as a family of locally unital algebra homomorphisms $\Phi_d : QH_d \to NT_d$ for all $d \geq 0$, such that $1_{\boldsymbol{i}} \mapsto 1_{\boldsymbol{i}}$ for each $\boldsymbol{i} \in I^d$ and

$$a_{d_2,d_1} \circ \Phi_{d_2} \otimes \Phi_{d_1} = \Phi_{d_2+d_1} \circ b_{d_2,d_1} \tag{6.13}$$

for all $d_1, d_2 \geq 0$, where $b_{d_2,d_1} : QH_{d_2} \otimes QH_{d_1} \to QH_{d_2+d_1}$ is the obvious embedding defined by horizontal concatenation of diagrams.

To construct Φ_d, we start from the monoidal superfunctor Ψ from Theorem 6.4. This induces superalgebra homomorphisms $\Psi_d : AHC_d \to \mathrm{End}(sF^d)$ for all $d \geq 0$, where $\mathrm{End}(sF^d)$ denotes supernatural endomorphisms of $sF^d : s\mathcal{O} \to s\mathcal{O}$. For each $M \in \mathrm{ob}\, s\mathcal{O}$, Corollary 3.3 implies that $\mathrm{ev}_M \circ \Psi_d : AHC_d \to \mathrm{End}_{s\mathcal{O}}(sF^d M)$ factors through all sufficiently large cyclotomic quotients $AHC_d(\mu)$. Hence, Ψ_d extends uniquely to a locally unital superalgebra homomorphism $\widehat{\Psi}_d : \widehat{AHC}_d \to SNT_d$, where

$$SNT_d := \bigoplus_{\boldsymbol{j},\boldsymbol{j}' \in J^d} \mathrm{Hom}(sF_{\boldsymbol{j}}, sF_{\boldsymbol{j}'}) \subset \mathrm{End}(sF^d)$$

and $sF_{\boldsymbol{j}} := sF_{j_d} \cdots sF_{j_1}$. Composing $\widehat{\Psi}_d$ with the isomorphism from Theorem 6.6 and the inclusion $QHC_d \hookrightarrow \widehat{QHC}_d$, we obtain a locally unital superalgebra homomorphism $\Theta_d : QHC_d \to SNT_d$. It is obvious from Definitions 6.1 and 6.5 that there is a locally unital algebra homomorphism in $: QH_d \to (QHC_d)_{\bar{0}}$ sending the idempotent $1_{\boldsymbol{i}}$ to $1_{\boldsymbol{j}}$ for $\boldsymbol{j}$ with $j_r := \sqrt{z+i_r}\sqrt{z+i_r+1}$, and taking the elements of $QH_d 1_{\boldsymbol{i}}$ defined by the dot on the rth strand and the crossing of the rth and $(r+1)$th strands to $\xi_r 1_{\boldsymbol{j}}$ and $\tau_r 1_{\boldsymbol{j}}$, respectively. Also, recalling (5.1), restriction from $s\mathcal{O}$ to $\mathcal{O}$ defines a homomorphism

$$\mathrm{pr} : \bigoplus_{\boldsymbol{j},\boldsymbol{j}' \in J_+^d} 1_{\boldsymbol{j}'} (SNT_d)_{\bar{0}} 1_{\boldsymbol{j}} \to NT_d$$

where $J_+ := \left\{ \sqrt{z+i}\sqrt{z+i+1} \,\middle|\, i \in I \right\} \subset J$. Then the composition $\mathrm{pr} \circ \Theta_d \circ \mathrm{in}$ gives us the desired locally unital homomorphism $\Phi_d : QH_d \to NT_d$ sending $1_{\boldsymbol{i}} \mapsto 1_{\boldsymbol{i}}$ for each $\boldsymbol{i} \in I^d$. It just remains to observe that the property (6.13) is satisfied, and that $\Phi_d(x_r 1_{\boldsymbol{i}})_M$ is nilpotent for each r, $\boldsymbol{i} \in I^d$ and $M \in \mathrm{ob}\, \mathcal{O}$. These things follow from the explicit formulae in Theorems 6.4 and 6.6 plus Corollary 3.3 once again. □

7. Proof of the Main Theorem

Everything is now in place for us to be able to prove the Main Theorem from the introduction. Recall $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ is a sign sequence, and $V^{\otimes \boldsymbol{\sigma}}$ denotes the $\mathfrak{sl}_\infty$-module $V^{\sigma_1} \otimes \cdots \otimes V^{\sigma_n}$. The following is a special case of [BLW, Definition 2.10], which reformulated [LW, Definiton 3.2] for tensor products of minuscule representations; it may be helpful to recall Definitions 4.1, 5.3 and 6.1 at this point.

Definition 7.1. An $\mathfrak{sl}_\infty$-*tensor product categorification* of $V^{\otimes \boldsymbol{\sigma}}$ is the following data:

- a highest weight category $\mathcal{C}$ with weight poset $(\mathbf{B}, \preceq)$;
- adjoint pairs (F_i, E_i) of endofunctors of $\mathcal{C}$ for each $i \in I$;
- a strict monoidal functor $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{C})$ with $\Phi(i) = F_i$ for each $i \in I$.

We impose the following additional axioms for all $i \in I$, $\boldsymbol{b} \in \mathbf{B}$ and $M \in \mathrm{ob}\, \mathcal{C}$:

- E_i is isomorphic to a left adjoint of F_i;
- $F_i \Delta(\boldsymbol{b})$ has a Δ-flag with sections $\{\Delta(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t) \mid 1 \leq t \leq n, i\text{-}\mathrm{sig}_t(\boldsymbol{b}) = \mathtt{f}\}$;
- $E_i \Delta(\boldsymbol{b})$ has a Δ-flag with sections $\{\Delta(\boldsymbol{b} - \sigma_t \boldsymbol{d}_t) \mid 1 \leq t \leq n, i\text{-}\mathrm{sig}_t(\boldsymbol{b}) = \mathtt{e}\}$;
- the endomorphism $\Phi\Big(\underset{i}{\bullet}\Big)_M : F_i M \to F_i M$ is nilpotent.

Theorems 5.1, 5.4, 5.6 and 6.2 together imply:

Theorem 7.2. *The supercategory $s\mathcal{O}$ defined in section 2 splits as $\mathcal{O} \oplus \Pi\mathcal{O}$, with $\mathcal{O}$ admitting all of the additional structure needed to make it into an $\mathfrak{sl}_\infty$-tensor product categorification of $V^{\otimes\boldsymbol{\sigma}}$.*

As we already mentioned in the introduction, to complete the proof of our Main Theorem, we just need to appeal to the following results from [BLW]:

Theorem 7.3. *The supercategory $s\mathcal{O}'$ from the introduction decomposes as $\mathcal{O}' \oplus \Pi\mathcal{O}'$, with $\mathcal{O}'$ admitting the structure of an $\mathfrak{sl}_\infty$-tensor product categorification of $V^{\otimes\boldsymbol{\sigma}}$.*

Proof. This is a special case of [BLW, Theorem 3.10]. □

Theorem 7.4. *Any two $\mathfrak{sl}_\infty$-tensor product categorifications of $V^{\otimes\boldsymbol{\sigma}}$ are strongly equivariantly equivalent (in the sense of* [LW, Definition 3.1]*).*

Proof. This is a special case of [BLW, Theorem 2.12]. □

We get at once that the categories $\mathcal{O}$ and $\mathcal{O}'$ are equivalent, hence so too are $s\mathcal{O}$ and $s\mathcal{O}'$. This already proves the Main Theorem from the introduction in the case that n is even. When n is odd, one also needs to apply the Lemma from the introduction to see that the supercategory $s\mathcal{O}$ in the statement of the Main Theorem is the Clifford twist of the supercategory $s\mathcal{O}$ being studied here.

8. Canonical basis

Combining our Main Theorem with the results of [CLW, BLW], it follows that the composition multiplicities $[M(\boldsymbol{a}) : L(\boldsymbol{b})]$ can be obtained by evaluating certain parabolic Kazhdan-Lusztig polynomials at $q = 1$. In this section, we explain a simple algorithm to compute these polynomials explicitly. The algorithm is similar in spirit to the algorithm explained in [B1, §2-j], but actually the variant here is both easier to implement and a little faster. It also has the advantage of working for arbitrary sign sequences $\boldsymbol{\sigma}$, whereas the approach in [B1] only makes sense for *normally-ordered* $\boldsymbol{\sigma}$'s, i.e. ones in which all $+$'s preceed all $-$'s. (But note that one can easily transition between different sign sequences as explained in [CL, §5].)

We first need to introduce the quantum analog of the $\mathfrak{sl}_\infty$-module $V^{\otimes\boldsymbol{\sigma}}$. We will use similar notation to before, but decorated with dots to indicate q-analogs. Consider the generic quantized enveloping algebra $U_q\mathfrak{sl}_\infty$ over the field $\mathbb{Q}(q)$. This has standard generators $\{\dot{e}_i, \dot{f}_i, \dot{k}_i, \dot{k}_i^{-1} \mid i \in I\}$. We work with the comultiplication Δ defined from

$$\Delta(\dot{f}_i) = 1 \otimes \dot{f}_i + \dot{f}_i \otimes \dot{k}_i, \quad \Delta(\dot{e}_i) = \dot{k}_i^{-1} \otimes \dot{e}_i + \dot{e}_i \otimes 1, \quad \Delta(\dot{k}_i) = \dot{k}_i \otimes \dot{k}_i. \tag{8.1}$$

We have the natural $U_q\mathfrak{sl}_\infty$-module $\dot{V}^+$ on basis $\{\dot{v}_i^+ \mid i \in I\}$, and its dual $\dot{V}^-$ on basis $\{\dot{v}_i^- \mid i \in I\}$. The Chevalley generators $\dot{f}_i$ and $\dot{e}_i$ act on these basis vectors by exactly the same formulae (4.1)–(4.2) as at $q = 1$, and also

$$\dot{k}_i \dot{v}_j^+ = q^{\delta_{i,j} - \delta_{i+1,j}} \dot{v}_j^+, \qquad \dot{k}_i \dot{v}_j^- = q^{\delta_{i+1,j} - \delta_{i,j}} \dot{v}_j^-. \tag{8.2}$$

Then we form the tensor space $\dot{V}^{\otimes\boldsymbol{\sigma}} := \dot{V}^{\sigma_1} \otimes \cdots \otimes \dot{V}^{\sigma_n}$, which is a $U_q\mathfrak{sl}_\infty$-module with its monomial basis $\left\{\dot{v}_{\boldsymbol{b}} := \dot{v}_{b_1}^{\sigma_1} \otimes \cdots \otimes \dot{v}_{b_n}^{\sigma_n} \,\middle|\, \boldsymbol{b} \in \mathbf{B}\right\}$.

Next we pass to a certain completion $\widehat{V}^{\otimes\boldsymbol{\sigma}}$. Recall the inverse dominance order $\succeq$ on P^n from Definition 4.1. We define $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ to be the $\mathbb{Q}(q)$-vector space consisting of formal linear combinations of the form $\sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}$ for rational functions $p_{\boldsymbol{b}}(q) \in \mathbb{Q}(q)$, such that the *support* $\{\boldsymbol{b} \in \mathbf{B} \mid p_{\boldsymbol{b}}(q) \neq 0\}$ is contained in a finite union of sets of the form $\{\boldsymbol{b} \in \mathbf{B} \mid \mathbf{wt}(\boldsymbol{b}) \succeq \boldsymbol{\beta}\}$ for $\boldsymbol{\beta} \in P^n$.

Lemma 8.1. *The action of $U_q\mathfrak{sl}_\infty$ on $\dot{V}^{\otimes\boldsymbol{\sigma}}$ extends to a well-defined action on $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ such that $u\big(\sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}\big) = \sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\, u\dot{v}_{\boldsymbol{b}}$ for every $u \in U_q\mathfrak{sl}_\infty$. Moreover, $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ splits as the direct sum of its weight spaces.*

Proof. For the first assertion, we need to show that the expression $\sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\, u\dot{v}_{\boldsymbol{b}}$ satisfies the condition on its support required to belong to $\widehat{V}^{\otimes\boldsymbol{\sigma}}$. It suffices to check this for $u \in \{\dot{f}_i, \dot{e}_i \mid i \in I\}$. If $\mathbf{wt}(\boldsymbol{b}) \succeq (\beta_1, \dots, \beta_n)$ and $v_{\boldsymbol{a}}$ appears with non-zero coefficient in the expansion of $\dot{e}_i v_{\boldsymbol{b}}$ (resp. $\dot{f}_i v_{\boldsymbol{b}}$), then $\mathbf{wt}(\boldsymbol{a})$ is equal to $\mathbf{wt}(\boldsymbol{b})$ with α_i added (resp. subtracted) from one of its entries. Hence, $\mathbf{wt}(\boldsymbol{a}) \succeq (\beta_1 + \alpha_i, \beta_2, \dots, \beta_n)$ (resp. $\mathbf{wt}(\boldsymbol{a}) \succeq (\beta_1, \dots, \beta_{n-1}, \beta_n - \alpha_i)$). This is all that is needed.

The fact that $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ is a direct sum of its weight spaces follows because all $v_{\boldsymbol{b}}$ with $\mathbf{wt}(\boldsymbol{b}) \succeq (\beta_1, \dots, \beta_n)$ are of the same weight $\beta_1 + \cdots + \beta_n$. $\square$

The completion $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ admits a *bar involution* $\psi : \widehat{V}^{\otimes\boldsymbol{\sigma}} \to \widehat{V}^{\otimes\boldsymbol{\sigma}}$ which is anti-linear with respect to the field automorphism $\mathbb{Q}(q) \to \mathbb{Q}(q), q \mapsto q^{-1}$. To define ψ, let Θ be Lusztig's quasi-R-matrix from [L, Theorem 4.1.2]; note for this due to our different choice of Δ compared to [L] that Lusztig's v is our q^{-1} (and his E_i, F_i, K_i are our $\dot{e}_i, \dot{f}_i, \dot{k}_i^{-1}$). We proceed by induction on n, setting $\psi(\dot{v}_i^+) = \dot{v}_i^+$ and $\psi(\dot{v}_i^-) = \dot{v}_i^-$ in case $n = 1$. For $n > 1$, let $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{b}}$ denote the $(n-1)$-tuples $(\sigma_1, \dots, \sigma_{n-1})$ and $(b_1, \dots, b_{n-1})$, respectively. Assuming that the analog $\bar{\psi}$ of ψ on the space $\widehat{V}^{\otimes\bar{\boldsymbol{\sigma}}}$ has already been defined by induction, we define ψ on $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ by setting

$$\psi\left(\sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}\right) := \sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q^{-1})\,\Theta\big(\bar{\psi}(\dot{v}_{\bar{\boldsymbol{b}}}) \otimes \dot{v}_{b_n}^{\sigma_n}\big). \tag{8.3}$$

Lemma 8.2. *The antilinear map ψ defined by (8.3) is a well-defined involution of $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ preserving all weight spaces and commuting with the actions of $\dot{f}_i, \dot{e}_i$ for all $i \in I$. Moreover, $\psi(\dot{v}_{\boldsymbol{b}})$ is equal to $\dot{v}_{\boldsymbol{b}}$ plus a (possibly infinite) $\mathbb{Z}[q, q^{-1}]$-linear combination of $\dot{v}_{\boldsymbol{a}}$'s for $\boldsymbol{a} \succ \boldsymbol{b}$.*

Proof. Recall that Θ is a formal sum of terms Θ_β for $\beta \in \bigoplus_{i\in I} \mathbb{N}\,\alpha_i$, with $\Theta_0 = 1$ and $\Theta_\beta \in (U_q^-\mathfrak{sl}_\infty)_{-\beta} \otimes (U_q^+\mathfrak{sl}_\infty)_\beta$. The only monomials in the generators of $U_q^+\mathfrak{sl}_\infty$ that are non-zero on $\dot{v}_j^{\sigma_n}$ are of the form $\dot{e}_i\dot{e}_{i+1}\cdots\dot{e}_{j-1}$ for $i \leq j$ if $\sigma_n = +$ (resp. the form $\dot{e}_{i-1}\dot{e}_{i-2}\cdots\dot{e}_j$ for $i \geq j$ if $\sigma_n = -$). Using also the integrality of the quasi-R-matrix from [L, Corollary 24.1.6] (or a direct calculation from [L, Theorem 4.1.2(b)]), it follows for any $v \in \widehat{V}^{\otimes\bar{\boldsymbol{\sigma}}}$ that

$$\Theta\left(v \otimes \dot{v}_j^{\sigma_n}\right) = v \otimes \dot{v}_j^{\sigma_n} + \sum_i (\Theta_{i,j} v) \otimes \dot{v}_i^{\sigma_n} \tag{8.4}$$

summing over $i < j$ if $\sigma_n = +$ (resp. $i > j$ if $\sigma_n = -$), for $\Theta_{i,j}$'s that are $\mathbb{Z}[q, q^{-1}]$-linear combinations of monomials obtained by multiplying the generators $\dot{f}_i, \dot{f}_{i+1}, \dots, \dot{f}_{j-1}$ (resp. $\dot{f}_{i-1}, \dot{f}_{i-2}, \dots, \dot{f}_j$) together in some order. By induction, $\bar{\psi}(\dot{v}_{\bar{\boldsymbol{b}}})$ equals $\dot{v}_{\bar{\boldsymbol{b}}}$ plus a $\mathbb{Z}[q, q^{-1}]$-linear combination of $\dot{v}_{\bar{\boldsymbol{a}}}$'s for $\bar{\boldsymbol{a}} \succ \bar{\boldsymbol{b}}$. Combining these two statements, we deduce that

$$\Theta\left(\bar{\psi}(\dot{v}_{\bar{\boldsymbol{b}}}) \otimes \dot{v}_{b_n}^{\sigma_n}\right) = \dot{v}_{\boldsymbol{b}} + (\text{a } \mathbb{Z}[q, q^{-1}]\text{-linear combination of } \dot{v}_{\boldsymbol{a}}\text{'s for } \boldsymbol{a} \succ \boldsymbol{b}).$$

This shows that the formula (8.3) makes $\psi(\dot{v}_{\boldsymbol{b}})$ into a well-defined element of $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ of the desired form. The formula (8.3) also makes sense for arbitrary sums $\sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}$ due to the interval-finiteness of the inverse dominance ordering on P^n. Finally, to see that ψ commutes with the actions of all $\dot{f}_i$ and $\dot{e}_i$, and that it is an involution, one argues as in [L, §27.3.1]. $\square$

Now we are in a position to apply "Lusztig's Lemma" as in the proof of [L, Theorem 27.3.2] to deduce for each $\boldsymbol{b} \in \mathbf{B}$ that there is a unique vector $\dot{c}_{\boldsymbol{b}} \in \widehat{V}^{\otimes\boldsymbol{\sigma}}$ such that

- $\psi(\dot{c}_{\boldsymbol{b}}) = \dot{c}_{\boldsymbol{b}}$;
- $\dot{c}_{\boldsymbol{b}} = \dot{v}_{\boldsymbol{b}} +$ a (possibly infinite) $q\,\mathbb{Z}[q]$-linear combination of $\dot{v}_{\boldsymbol{a}}$'s for $\boldsymbol{a} \succ \boldsymbol{b}$.

This defines the *canonical basis* $\{\dot{c}_{\boldsymbol{b}} \,|\, \boldsymbol{b} \in \mathbf{B}\}$. It is known (but non-trivial) that each $\dot{c}_{\boldsymbol{b}}$ is always a *finite* sum of $\dot{v}_{\boldsymbol{a}}$'s, i.e. $\dot{c}_{\boldsymbol{b}} \in \dot{V}^{\otimes\boldsymbol{\sigma}}$ before completion. Moreover, the polynomials $d_{\boldsymbol{a},\boldsymbol{b}}(q)$ arising from the expansion

$$\dot{c}_{\boldsymbol{b}} = \sum_{\boldsymbol{a}\in\mathbf{B}} d_{\boldsymbol{a},\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{a}} \tag{8.5}$$

are some finite type A parabolic Kazhdan-Lusztig polynomials (suitably normalized). All of these statements have a natural representation theoretic explanation discussed in detail in [BLW, §5.9]. In particular, the results of [BLW] (or [CLW]) imply the following.

Theorem 8.3. *Under the identification of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{O}^{\Delta})$ with $V^{\otimes\boldsymbol{\sigma}}$ from Theorem 5.6, $[P(\boldsymbol{b})]$ corresponds to the specialization $c_{\boldsymbol{b}}$ of the canonical basis element $\dot{c}_{\boldsymbol{b}}$ at $q = 1$. Equivalently, $(P(\boldsymbol{b}) : M(\boldsymbol{a})) = [M(\boldsymbol{a}) : L(\boldsymbol{b})] = d_{\boldsymbol{a},\boldsymbol{b}}(1)$ for each $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}$.*

We are ready to explain our new algorithm to compute $\dot{c}_{\boldsymbol{b}}$. We proceed by induction on n. In case $n = 1$, we have that $\dot{c}_{\boldsymbol{b}} = \dot{v}_{\boldsymbol{b}}$ always. If $n > 1$, we first compute $\dot{c}_{\bar{\boldsymbol{b}}} \in \widehat{V}^{\otimes\bar{\boldsymbol{\sigma}}}$. It is a linear combination of finitely many $\dot{v}_{\bar{\boldsymbol{a}}}$'s for $\bar{\boldsymbol{a}} \succeq \bar{\boldsymbol{b}}$. Then we define $j \in I$ as follows.

- If $\sigma_n = +$ then j is the greatest integer such that $j \leq b_n$, and the following hold for all $1 \leq r < n$ and all tuples $\bar{\boldsymbol{a}} = (a_1, \dots, a_{n-1})$ such that $\dot{v}_{\bar{\boldsymbol{a}}}$ occurs in the expansion of $\dot{c}_{\bar{\boldsymbol{b}}}$:
  - if $\sigma_r = +$ then $j \leq a_r$;
  - if $\sigma_r = -$ then $j < a_r$.
- If $\sigma_n = -$ then j is the smallest integer such that $j \geq b_n$, and the following hold for all $1 \leq r < n$ and all tuples $\bar{\boldsymbol{a}} = (a_1, \dots, a_{n-1})$ such that $\dot{v}_{\bar{\boldsymbol{a}}}$ occurs from the expansion of $\dot{c}_{\bar{\boldsymbol{b}}}$:
  - if $\sigma_r = -$ then $j \geq a_r$;
  - if $\sigma_r = +$ then $j > a_r$.

Lemma 8.4. *In the above notation, we have that $\Theta\left(\dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j^{\sigma_n}\right) = \dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j^{\sigma_n}$.*

Proof. By (8.4), we have that $\Theta\left(\dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j^{\sigma_n}\right) = \dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j^{\sigma_n} + \sum_i \left(\Theta_{i,j}\dot{c}_{\bar{\boldsymbol{b}}}\right) \otimes \dot{v}_i^{\sigma_n}$ summing over $i < j$ if $\sigma_n = +$ (resp. $i > j$ if $\sigma_n = -$), where $\Theta_{i,j}$ is a linear combination of non-trivial monomials in the generators $\dot{f}_{j-1}, \dot{f}_{j-2}, \dots, \dot{f}_i$ (resp. $\dot{f}_j, \dot{f}_{j+1}, \dots, \dot{f}_{i-1}$). By the definition of j, all of these generators act as zero on $\dot{c}_{\bar{\boldsymbol{b}}}$. $\square$

Lemma 8.4 shows that the vector $\dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j^{\sigma_n} \in \widehat{V}^{\otimes\boldsymbol{\sigma}}$ is fixed by ψ. Hence, so too is $X\left(\dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j^{\sigma_n}\right)$ where

$$X := \begin{cases} \dot{f}_{b_n-1} \cdots \dot{f}_{j+1}\dot{f}_j & \text{if } \sigma_n = +, \\ \dot{f}_{b_n} \cdots \dot{f}_{j-2}\dot{f}_{j-1} & \text{if } \sigma_n = -. \end{cases}$$

By Lemma 4.3, this new vector equals $\dot{v}_{\boldsymbol{b}}$ plus a $\mathbb{Z}[q,q^{-1}]$-linear combination of $\dot{v}_{\boldsymbol{a}}$'s for $\boldsymbol{a} \succ \boldsymbol{b}$. If all but its leading coefficient lie in $q\,\mathbb{Z}[q]$, it is already the desired vector $\dot{c}_{\boldsymbol{b}}$. Otherwise, one picks $\boldsymbol{a} \succ \boldsymbol{b}$ minimal so that the $\dot{v}_{\boldsymbol{a}}$-coefficient is not in $q\,\mathbb{Z}[q]$, then subtracts a bar-invariant multiple of the recursively computed vector $\dot{c}_{\boldsymbol{a}}$ to remedy this defficiency. Continuing in this way, we finally obtain a bar-invariant vector with all of the required properties to be $\dot{c}_{\boldsymbol{b}}$.

The algorithm just described has been implemented in GAP, and is available at `http://pages.uoregon.edu/brundan/papers/A.gap`. It is not obvious to us that it terminates in finite time for every $\boldsymbol{b} \in \mathbf{B}$. Based on examples, we believe that this is indeed the case.

Example 8.5. Suppose $\boldsymbol{\sigma} = (+,+,-,-)$ and $\boldsymbol{b} = (1,2,2,1)$. By induction, we have $\dot{c}_{(1,2,2)} = \dot{v}_{(1,2,2)} + q\dot{v}_{(2,1,2)} + q\dot{v}_{(1,3,3)} + q^2\dot{v}_{(3,1,3)}$, so take $j = 4$. We compute

$$\begin{aligned}\dot{f}_1\dot{f}_2\dot{f}_3(\dot{v}_{(1,2,2,4)} + q\dot{v}_{(2,1,2,4)} + q\dot{v}_{(1,3,3,4)} + q^2\dot{v}_{(3,1,3,4)}) &= \dot{v}_{(1,2,2,1)} + q\dot{v}_{(1,2,1,2)} + q\dot{v}_{(2,1,2,1)}\\ &+ q\dot{v}_{(1,4,1,4)} + q\dot{v}_{(1,3,3,1)} + q^2\dot{v}_{(2,1,1,2)} + q^2\dot{v}_{(3,1,3,1)} + q^2\dot{v}_{(2,3,3,2)} + q^2\dot{v}_{(4,1,1,4)}\\ &+ q^2\dot{v}_{(2,4,2,4)} + (1+q^2)\dot{v}_{(1,3,1,3)} + q^3\dot{v}_{(3,2,3,2)} + q^3\dot{v}_{(4,2,2,4)} + (q+q^3)\dot{v}_{(2,3,2,3)}\\ &+ (q+q^3)\dot{v}_{(3,1,1,3)} + (q+q^3)\dot{v}_{(2,2,2,2)} + (q^2+q^4)\dot{v}_{(3,2,2,3)}.\end{aligned}$$

Then we subtract the recursively computed

$$\begin{aligned}\dot{c}_{(1,3,1,3)} = \dot{v}_{(1,3,1,3)} &+ q\dot{v}_{(3,1,1,3)} + q\dot{v}_{(2,3,2,3)} + q\dot{v}_{(1,4,1,4)}\\ &+ q^2\dot{v}_{(3,2,2,3)} + q^2\dot{v}_{(4,1,1,4)} + q^2\dot{v}_{(2,4,2,4)} + q^3\dot{v}_{(4,2,2,4)}\end{aligned}$$

to get $\dot{c}_{(1,2,2,1)}$.

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