WHAT IS MY RESEARCH ALL ABOUT?

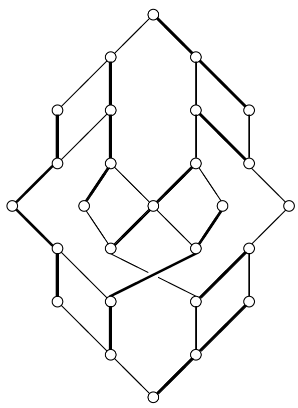
JONATHAN BRUNDAN

My research spans two different branches of mathematics: *representations of finite groups* and *Lie theory*. Actually much of my time is spent trying to understand the *relationships* between the two. This is a common theme in mathematics: by playing off two different — but connected — areas of mathematics against each other you can often make progress in understanding them both.

To illustrate the idea, I will try to describe two specific aspects of my work: the notion of *crystal* from Lie theory, and certain *branching rules* in group representation theory. It came as a great surprise to mathematicians when a connection between these two things was first noticed, around 1995 — but we only recently have a satisfactory proof explaining the coincidence.

1. Crystals

So the first piece of mathematics I need to introduce is the notion of a crystal. This is nothing to do with the familiar sort of crystals that appear in science, but the name is a good one as I shall try to explain in a while. Probably the best thing is to see an example right away:



Notice that this crystal consists of a bunch of dots ("vertices") connected by lines ("edges"). The edges are meant to be of two different colors: I have drawn them either thin or thick accordingly.

The theory of crystals was discovered around 1990 by two great modern mathematicians, G. Lusztig and M. Kashiwara. I often tell people that the notion of crystal is the single most significant advance in Lie theory since the time of the mathematician/physicist Hermann Weyl, and I think I am only exaggerating slightly! Crystals come in families, for instance, there are A_n-, B_n-, C_n-, D_n- E_6-, E_7-, E_8-, F_4- and G_2-crystals. The suffix indicates the number of colors needed to color the edges of the crystal. (The example above happens to be an A_2-crystal, so just two colors are needed for its edges.)

A mathematician or physicist would recognize the symbols $A_n, B_n, \dots$ as the names of the *simple Lie groups*. A group in mathematics means a specified bunch of "elements" together with a well-behaved rule for "multiplying" two elements together to get a third. The simple Lie groups are absolutely central objects since they describe the basic symmetries of many sorts of diverse structures in mathematics and physics. For instance, B_n is the group with elements being the symmetries of a sphere in $(2n + 1)$-dimensional space. If you take two such symmetries and do them one after the other, you get a third "composite" symmetry of the sphere. This is the rule for "multiplying" symmetries together, which makes the set of symmetries of a sphere into a group.

To continue the discussion, let us just consider for a moment the case of A_2. This Lie group can be realized concretely as the set of all 3×3 matrices of determinant 1 under matrix multiplication. It contains the elements

$$\begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix},$$

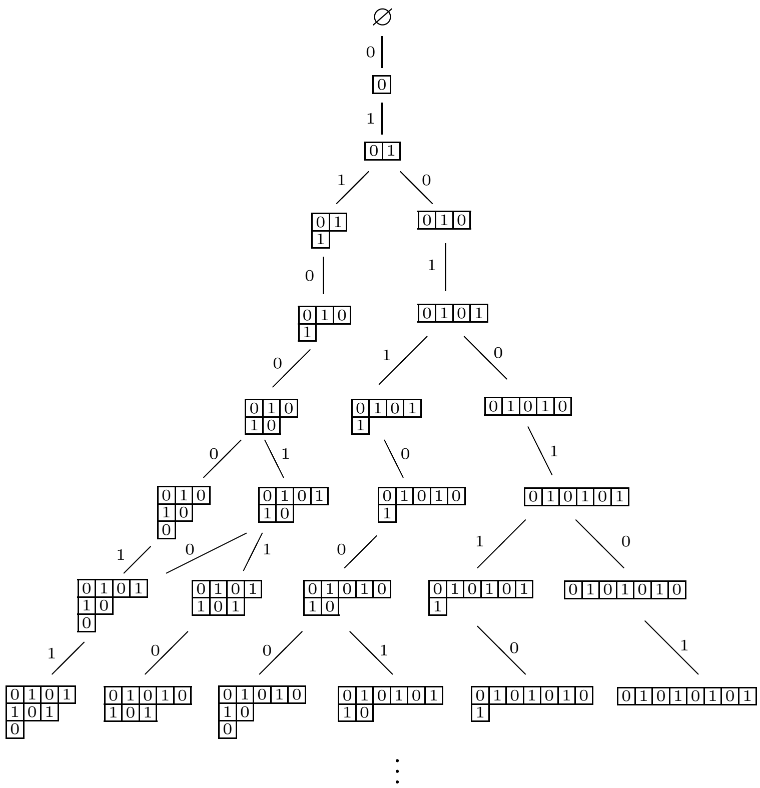
where c stands for an arbitrary complex number. In fact, you get all elements of the group A_2 (in this realization) just by starting with the special matrices listed above and multiplying them together in all possible ways.

Whenever a group is realized as a bunch of matrices, like above, we have what is known as a *representation* of the group. A given group has very many different representations in terms of matrices. For example the group A_2 has another representation by 27×27 matrices.

How can I write this 27 dimensional representation down explicitly? One way would be to write down a list of 27×27 matrices, corresponding *in the new representation* to the 3×3 matrices listed above. This would be rather laborious, and to be honest is not all that useful in practice: the problem is that there is *too much* information in the explicit matrices. This is where the theory of crystals comes in. The crystal drawn above is the crystal corresponding to this 27 dimensional representation (notice at least that it has 27 vertices). It does not contain *all* of the information in the representation, rather it is a beautiful summary containing exactly the right amount of information needed for all sorts of practical purposes.

Why the name "crystal"? The construction of the crystal corresponding to a given representation of a Lie group depends fundamentally on the theory of *quantum groups*. These are deformations of Lie groups, with deforming parameter usually denoted q. In the *classical limit* $q \to 1$, the quantum group – in its given representation – specializes to the original Lie group. But there is a new possibility: the *non-classical limit* $q \to 0$ also makes sense. In this limit, the representation "crystallizes" out into the combinatorial structure that is its corresponding crystal (this is rather informal, but that is the idea!).

Let me give one more family of examples of crystals, that play a particularly important role in my work. These are the crystals of type $A_n^{(1)}$, and have infinitely many vertices, unlike the ones above. Here for example is the one of type $A_1^{(1)}$:



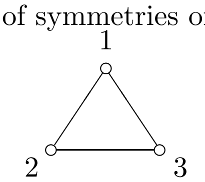
The edges in this crystal are of two different colors, which I have indicated this time by the numbers '0' and '1'. The vertices are labelled now by strange looking diagrams called *Young diagrams* — but you should ignore these for the time being and just pretend they are dots like before, since they don't really carry any of the structure. A good way to think of this crystal is as a series of increasing "energy levels". At the top, the symbol ∅ is the "vacuum vector" of zero energy. Then each level you go down in the picture – when an extra box is added to the Young diagrams – you step up an energy level. Energy levels go up for ever, though only the first few levels are shown in the picture.

There is actually a precise rule describing how to build each successive level from the previous one. But it is a very subtle pattern, and would be very hard to guess without being lead to it by the mathematics. The rule was first described by the Japanese mathematicians Misra and Miwa in 1992.

2. Branching rules

Now I am going to switch to discussing the representation theory of the *symmetric group*. Let me repeat: the representation theory of the symmetric group is a quite different area of mathematics to the Lie theory described above, but we have stumbled upon a remarkable and unexpected connection between the two areas which sheds light on both.

The symmetric group depends on a number n, and is usually denoted S_n. Like the Lie groups of §2.1, S_n can be though of as the symmetries of some geometric object, but this time the object has just *finitely many* symmetries. For example, S_3 is the group of symmetries of an equilateral triangle.



There are six such symmetries:

(1) a reflection in the vertical line swaps vertices 2 and 3 leaving 1 fixed;
(2) another reflection swaps 1 and 2 leaving 3 fixed;
(3) another reflection swaps 1 and 3 leaving 2 fixed;
(4) a counterclockwise rotation through 120° sends 1 to 2, 2 to 3 and 3 to 1 (I finally stopped saying the British 'anticlockwise' to the amusement of my students!);
(5) a clockwise rotation through 120° sends 1 to 3, 3 to 2 and 2 to 1;
(6) and the one you might forget: the trivial symmetry leaving everything where it is.

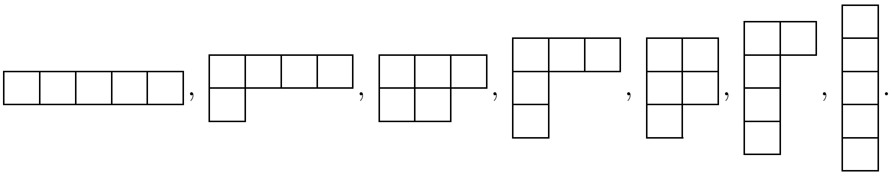
Next S_4 is the group of symmetries of the regular tetrahedron (also of the methane molecule!): there are 24 such symmetries. In general, S_n is the group of $n!$ symmetries of a regular simplex in $(n-1)$-dimensional space; this is the analogue of the tetrahedron in higher dimensions.

Now again we will be interested in *representations* of the symmetric group, i.e. ways of realizing this group as a bunch of matrices. Here for example is a 2 dimensional representation of the symmetric group S_3, in the form of a list of 6 matrices, one for each of the six elements listed above.

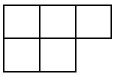
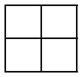
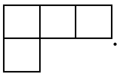
$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact, each S_n has a "natural" $(n-1)$-dimensional representation coming just from the way it was defined as symmetries of a simplex in $(n-1)$-dimensional space.

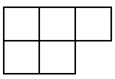
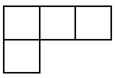
But again there are many other more complicated representations. They were classified completely by the German mathematician Frobenius at the beginning of the 20th century. In particular, he labelled the representations by *Young diagrams.* Roughly speaking, Young diagrams for S_n are n square boxes set together in such a way that they form a staircase shape. For example, the Young diagrams for S_5 are:



There are seven of them, corresponding to the seven irreducible representations of S_5.

Frobenius' famous student Schur studied the representations further and addressed the following question: what will happen if we restrict an irreducible representation of S_n to its natural subgroup S_{n-1}? Since S_{n-1} is smaller than S_n, the restriction will in general split into several irreducible representations of S_{n-1}. The rule which explains what happens is called the *branching rule*, and is fundamental for the whole theory. It says that one should remove exactly one box from the corresponding Young diagram in all possible ways. For example, the restriction of the representation corresponding to  splits into representations corresponding to  and .

The results of Frobenius and Schur described above are true only if the matrices are taken over the complex numbers. However it is also important to consider representations over more general fields of numbers, in particular fields of so-called *characteristic p*. Here the subject is far more complicated, and it was only in 1975 that the irreducible representations were even classified by Gordon James. The analogue of Schur's branching rules were not known until 1995.

Take a look back now at the picture of the crystal of type $A_1^{(1)}$. The vertices are labelled by Young diagrams, and in fact it is exactly these diagrams on the nth energy level that parametrize the irreducible representations of S_n in characteristic 2. *The edges describe the branching rules from S_n to S_{n-1} for the corresponding representations in characteristic* 2. For example the representation labelled by the diagram  restricts *irreducibly* just to the representation labelled by  — a quite different behaviour in characteristic 2 to Schur's branching rule over the complex numbers.

I want to emphasize is how much more subtle the branching rule is in characteristic p compared to Schur's case — after all it took 94 years for the generalization to be discovered. The breakthrough was made by Alexander

Kleshchev (also at the University of Oregon). I then generalized his work in my PhD thesis in 1996. It was initially just a coincidence that the graph describing the branching rules was *the same* as the crystal that had been worked out in 1992 by Lie theorists. But we now understand that this coincidence is just the tip of an iceberg: the two subjects are intimately connected in the most beautiful way.