6. Root datums and reductive algebraic groups

The main goal in this chapter is to give a sketch of how the classification of reductive algebraic groups goes. I really just want to give you the precise statement and some examples, but am going to skip very many details. This is a non-trivial theorem that would take a substantial amount of work to develop with full proofs...

6.1. **Tori and root datums.** Let us start by talking about *tori*. Recall an n-dimensional torus is an algebraic group isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$. For example, the subgroup D_n of GL_n consisting of all diagonal matrices is an n dimensional torus. Let T be an n-dimensional torus. The character group

$$X(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)^{\oplus n} \cong \mathbb{Z}^n.$$

An important point is that, given any two tori T and T',

$$\mathrm{Hom}(T, T') \cong \mathrm{Hom}(X(T'), X(T)).$$

So any homomorphism $f : X(T') \to X(T)$ of abelian groups induces a unique morphism $T \to T'$ of algebraic groups, and vice versa. To be fancy, you can view $X(?)$ as a contravariant equivalence of categories between the category of tori and the category of finitely generated free abelian groups.

All elements of a torus T are semisimple. So if V is any finite dimensional representation of T, every element of T is diagonalizable in its action on V by the Jordan decomposition. Moreover, they commute, hence we can actually diagonalize

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda$$

where

$$V_\lambda = \{v \in V \mid tv = \lambda(t)v \text{ for all } t \in T\}.$$

As before, the V_λ's are called the *weight spaces* of V with respect to the torus T.

Now let G be an arbitrary connected algebraic group. A *maximal torus* of G is what you'd think: a closed subgroup T that is maximal subject to being a torus.

Now start to assume that G is a reductive algebraic group. Let T be a maximal torus. Let $\mathfrak{g}$ be the Lie algebra of G. We can view $\mathfrak{g}$ as a representation of T via the adjoint action. It turns out moreover – using for the first time that G is reductive – that the zero weight space of $\mathfrak{g}$ with respect to T is exactly the Lie algebra $\mathfrak{t}$ of T itself. So we can decompose

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where Φ is the set of all $0 \neq \alpha \in X(T)$ such that the T-weight space $\mathfrak{g}_\alpha \neq 0$. You can already see the root system emerging... The difference now however is that the set Φ of roots is a subset of the free abelian group $X(T)$. Now using the assumption that G is reductive again, you show:

(1) Each $\mathfrak{g}_\alpha$ is one dimensional, and $\alpha \in \Phi$ iff $-\alpha \in \Phi$.
(2) The group $W = N_G(T)/T$ is a finite group that acts naturally on $X(T)$ and permutes the subset $\Phi \subseteq X(T)$.
(3) Let Q be the *root lattice*, the subgroup of $X(T)$ generated by Φ, and let $E = \mathbb{R} \otimes_{\mathbb{Z}} Q$. Fix a positive definite inner product on E that is invariant under the action of W. Then, (E, Φ) is an abstract root system.

We've now built out of G a root system (E, Φ), and realized the Weyl group W explicitly as the quotient group $N_G(T)/T$. Moreover, Φ is a subset of the character group $X(T)$ of T. I must stress that all these things take work to prove – it is usually harder than in the Lie algebra case. BUT everything works in arbitrary characteristic.

If G is semisimple, then G is determined up to isomorphism by its root system (E, Φ) together with the extra information given by the *fundamental group* $X(T)/Q$. However this is not the most natural point of view to classify the *reductive*, not just semisimple, groups. This is harder, since $X(T)$ will in general be of bigger rank than Q, and so there is much more freedom not captured by the fundamental group alone...

Let's prepare the way to state the classification of reductive algebraic groups in general. Let G be a reductive algebraic group, and let T be a maximal torus. Let $\Phi \subset X(T)$ be the root system of G, defined from the decomposition of $\mathfrak{g}$ as above. Let

$$X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$$

be the character group of T, and let

$$Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

be the cocharacter group. This is also a free abelian group of rank $\dim T$. Moreover, there is a pairing

$$X(T) \times Y(T) \to \mathbb{Z}$$

defined as follows. Given $\lambda \in X(T)$ and $\phi \in Y(T)$, the composite $\lambda \circ \phi$ is a map $\mathbb{G}_m \to \mathbb{G}_m$. So since $\mathrm{Aut}(\mathbb{G}_m) = \mathbb{Z}$,

$$(\lambda \circ \phi)(x) = x^{\langle \lambda, \phi \rangle}$$

for a unique $\langle \lambda, \phi \rangle \in \mathbb{Z}$.

For each $\alpha \in \Phi$, you prove that there is a (unique up to scalars) homomorphism

$$x_\alpha : \mathbb{G}_a \to G$$

such that

$$t x_\alpha(c) t^{-1} = x_\alpha(\alpha(t)c)$$

for all $c \in \mathbb{G}_a, t \in T$, such that the tangent map

$$dx_\alpha : L(\mathbb{G}_a) \to \mathfrak{g}_\alpha$$

is an isomorphism. Moreover, the x_α's can be normalized so that there is a homomorphism

$$\phi_\alpha : SL_2 \to G$$

such that

$$\phi_\alpha \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = x_\alpha(c), \phi_\alpha \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = x_{-\alpha}(c).$$

Define

$$\alpha^\vee : \mathbb{G}_m \to T, \alpha^\vee(c) = \phi_\alpha \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}.$$

So $\alpha^\vee \in Y(T)$. This is called the *coroot* associated to the root $\alpha \in \Phi$.

Now we have built a datum $(X(T), \Phi, Y(T), \Phi^\vee)$, where $\Phi^\vee$ is the set of all coroots. This is the *root datum* of G with respect to the torus T. (Actually, since all maximal tori in G are conjugate, it doesn't depend up to isomorphism on the choice of T.) The notion of root datum is the appropriate generalization of root system to take care of arbitrary reductive algebraic groups, not just the semisimple ones.

Here is an axiomatic formulation of the notion of root datum: a root datum is a quadruple $(X, \Phi, Y, \Phi^\vee)$ where

(a) X ("characters") and Y ("cocharacters") are free abelian groups of finite rank, in duality by a pairing $\langle ., . \rangle : X \times Y \to \mathbb{Z}$;
(b) $\Phi \subset X$ ("roots") and $\Phi^\vee \subset Y$ ("coroots") are finite subsets, and there is a given bijection $\alpha \mapsto \alpha^\vee$ from Φ to $\Phi^\vee$.

To record the additional axioms, define for $\alpha \in \Phi$ the endomorphisms $s_\alpha, s_\alpha^\vee$ of X, Y respectively by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee.$$

Then we have the axioms:

(RD1) For $\alpha \in \Phi$, $\langle \alpha, \alpha^\vee \rangle = 2$.
(RD2) For $\alpha \in \Phi$, $s_\alpha \Phi = \Phi$, $s_\alpha^\vee \Phi^\vee = \Phi^\vee$.

The datum $(X(T), \Phi, Y(T), \Phi^\vee)$ built from our algebraic group G earlier is such a gadget.

There is a notion of morphism of root datum

$$(X, \Phi, Y, \Phi^\vee) \to (X', \Phi', Y', (\Phi')^\vee):$$

a map $f : X' \to X$ that maps Φ' bijectively onto Φ and such that the dual map $f^\vee : Y \to Y'$ maps $f(\alpha)^\vee$ to $\alpha^\vee$ for all $\alpha \in \Phi'$. Hence there is a notion of isomorphism of root datums.

Now suppose that G, G' are reductive algebraic groups with maximal tori T, T' respectively and corresponding root data $(X(T), \Phi, Y(T), \Phi^\vee)$ and the primed version. Let $f : (X(T), \dots) \to (X'(T), \dots)$ be a morphism of root data. It induces a dual map $f : T \to T'$ of tori. The key step in the proof of the classification theorem is to show that f can be extended to a homomorphism $\bar{f} : G \to G'$.

Using it you prove in particular the *isomorphism theorem*:

Theorem 6.1. *Two reductive algebraic groups G, G' are isomorphic if and only if their root datums (relative to some maximal tori) are isomorphic.*

There is also an *existence theorem*:

Theorem 6.2. *For every root datum, there exists a corresponding reductive algebraic group G.*

Finally, one intriguing thing: given a root datum $(X, \Phi, Y, \Phi^\vee)$ there is the *dual* root datum $(Y, \Phi^\vee, X, \Phi)$. If G is a reductive algebraic group with root datum $(X, \Phi, Y, \Phi^\vee)$ you see there is a *dual group* $G^\vee$ with the corresponding dual root datum. Note the process of going from G to $G^\vee$ is very clumsy: I don't think there is any direct way of constructing the dual group out of the original.

Example 6.3. Suppose that G is a semisimple algebraic group. Let $Q = \mathbb{Z}\Phi \subset X(T)$. Here, Q and $X(T)$ have the same rank, so Q is a lattice in $X(T)$, and $X(T)/Q$ is a finite group, the fundamental group. Let P be the dual lattice to Q. Fixing a positive definite W-invariant inner product on $E = \mathbb{R} \otimes_{\mathbb{Z}} Q$, we can identify P with the weight lattice of the root system of G, and then everything is determined by the relationship between $Q \subseteq X(T) \subseteq P$. You can formulate the classification just of the *semisimple* algebraic groups in these terms.

Example 6.4. Let G be a semisimple algebraic group, and suppose that $Q \subseteq X(T) \subseteq P$ are as in the previous example. If $X(T) = P$, then G is called the *simply-connected* group of type Φ. If $X(T) = Q$, then G is called the *adjoint* group of this type. Now let G_{sc} be the simply-connected one, G_{ad} be the adjoint one. Let G be any other semisimple group of type Φ. Then, there is an inclusion $X(T) \hookrightarrow P = X(T_{sc})$. This induces a map $G_{sc} \twoheadrightarrow G$. Similarly, there is always a map $G \twoheadrightarrow G_{ad}$.

Example 6.5.
(1) Consider the root datum of GL_2. Here, $X(T)$ has basis ϵ_1, ϵ_2, these being the characters picking out the diagonal entries. Moreover, the positive root is $\alpha = \epsilon_1 - \epsilon_2$. Also $Y(T)$ has basis $\epsilon_1^\vee, \epsilon_2^\vee$, the dual basis, mapping $\mathbb{G}_m$ into each of the diagonal slots. The coroot is $\alpha^\vee = \epsilon_1^\vee - \epsilon_2^\vee$.
(2) GL_2 is its own dual group.
(3) Consider the root datum of $SL_2 \times \mathbb{G}_m$. Here, $X(T)$ has basis $\alpha/2, \epsilon$, $Y(T)$ has the dual basis $\alpha^\vee, \epsilon^\vee$ (here α is the usual positive root of SL_2).
(4) Consider the root datum of $PSL_2 \times \mathbb{G}_m$. Here, $X(T)$ has basis α, ϵ, $Y(T)$ has the dual basis $\alpha^\vee/2, \epsilon$. So $PSL_2 \times \mathbb{G}_m$ is the dual group to $SL_2 \times \mathbb{G}_m$.

Exercise 6.6.
(7) As an exercise in applying the classification, you can show that (1),(3) and (4) plus one more, the 4 dimensional torus, are *all* the reductive algebraic groups of dimension 4.

(8) The dual group to SL_n is PSL_n. The dual group to Sp_{2n} is SO_{2n+1}. The dual group to PSp_{2n} is $Spin_{2n+1}$. The dual group to SO_{2n} is SO_{2n}. The dual group to $Spin_{2n}$ is PSO_{2n}.

For more explicit constructions of root datums, see Springer, 7.4.7.

6.2. Complete varieties and the Borel fixed point theorem.

Definition 6.7. A variety X is called *complete* if for all varieties Y the projection

$$pr_Y : X \times Y \to Y$$

is a closed map.

This is the analogue of compactness in algebraic geometry. Here is an example of a space that is *not* complete:

Example 6.8. $\mathbb{A}^1$ is not complete. For consider the projection map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, (x, y) \mapsto y$. It sends the closed subvariety $\{(x, y) \mid xy = 1\}$ to $\mathbb{A}^1 - \{0\}$ which is not closed.

Theorem 6.9. *$\mathbb{P}^n$ is complete.*

Proof. We need to show for any variety Y that the projection morphism

$$\pi : \mathbb{P}^n \times Y \to Y$$

is closed. It suffices to deal with the case that Y is affine and irreducible. Put $A = k[Y]$, $S = A[T_0, \dots, T_n]$. We can view S as an algebra of functions on $k^{n+1} \times Y$. If I is a proper homogeneous ideal in S put

$$V(I) = \{([x], y) \in \mathbb{P}^n \times Y \mid f(x, y) = 0 \text{ for all } f \in I\}.$$

where $[x]$ denotes the point of $\mathbb{P}^n$ defined by $x \in k^{n+1} - \{0\}$. You should remember we looked at something like this when we described explicitly the closed sets in $\mathbb{P}^n$: they were the common zeros of proper homogeneous ideals of $k[T_0, \dots, T_n]$. In the new situation you show similarly:

- the closed sets in $\mathbb{P}^n \times Y$ are of the form $V(I)$ for proper homogeneous ideals in S;
- $V(I) = \varnothing$ if and only if $\sqrt{I} = (T_0, \dots, T_n)$;
- $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal.

Now we have to show that π maps closed sets to closed sets. Its enough to show it maps closed irreducible sets to closed irreducible sets (since any closed set is a finite union of irreducible closed sets). Thus we have to show all the irreducible sets $\pi V(I)$ are closed for all proper prime homogeneous ideals I in S. Let Y_0 be the closure of $\pi V(I)$, also irreducible and affine. Then $V(I)$ is contained in $\mathbb{P}^n \times Y_0$ and $\pi : V(I) \to Y_0$ is dominant. Replacing Y by Y_0 this reduces to showing that $\pi V(I) = Y$ for all proper prime homogeneous ideals I in S such that $\pi : V(I) \to Y$ is dominant. Note that $V(I) \subseteq \mathbb{P}^n \times V(A \cap I)$ so $\pi V(I) \subseteq V(A \cap I)$. So the dominance assumption on π means $V(A \cap I) = Y$, i.e. $A \cap I = \{0\}$.

We are now reduced to showing the following: given a proper prime homogeneous ideal I in S such that $A \cap I = \{0\}$ we want to prove for every $y \in Y$ that there exists $[x] \in \mathbb{P}^n$ with $([x], y) \in V(I)$. Let M be the maximal ideal of A of all functions vanishing at y. Then $J = MS + I$ is a proper homogeneous ideal in S. We have to prove that $V(J) \neq \varnothing$ (because if $([x], y') \in V(J)$ then pick i so $T_i(x) \neq 0$ and consider $fT_i \in J$ for $f \in M$: its value on (x, y') is zero hence $f(y') = 0$. This is for all $f \in M$ which implies $y' = y$.)

Therefore assume for a contradiction that I is a proper prime homogeneous ideal in S such that $A \cap I = \{0\}$, $y \in Y$ is a point with maximal ideal M in A, and $V(J) = \varnothing$ where $J = MS + I$. This means that $\sqrt{J} = (T_0, \dots, T_n)$, so there is some $h > 0$ such that $T_0^h, \dots, T_n^h \in J$. Equivalently there is some $l > 0$ such that the set $S_l \subset S$ of homogeneous polynomials of degree l lies in J. Put $N = S_l / S_l \cap I$. This is a finitely generated A-module. Our assumptions imply that every element of S_l looks like $\sum a_i b_i + j$ for $a_i \in M, b_i \in S_l$ and $j \in S_l \cap I$. Hence $N = MN$. Hence by Nakayama's lemma there's some element $a \in A$ with $aN = \{0\}$ and $a - 1 \in M$. It follows that $a \notin I$ (otherwise $J = S$) and $aS_l \subseteq I$. Since I is a prime ideal we deduce that $S_l \subseteq I$, i.e. $N = \{0\}$. But this means that $I \subset S_l$ so $V(I) = \varnothing$ which is a contradiction (Y would be empty...). □

The following is a rather easy exercise:

Proposition 6.10. (i) *If X is complete and Y is closed in X then Y is complete.*
(ii) *If X, Y are complete so is $X \times Y$.*
(iii) *If $\phi : X \to Y$ is a morphism and X is complete, then $\phi(X)$ is closed and complete.*
(iv) *If Y is a complete subvariety of X, then Y is closed.*
(v) *If X is complete and affine, then X is finite.*

(For (i) and (ii) its easy from the definition.

For (iii) consider the graph $(x, \phi(x))$ which is closed in $X \times Y$. You get easily that $\phi(X)$ is closed in Y from this. Its also complete which you see by considering $\phi \times 1 : X \times Z \to \phi(X) \times Z$...

For (iv) apply (iii) to the inclusion of Y into X.

For (v) let X be irreducible, complete and affine. Suppose $f : X \to \mathbb{A}^1$ is a morphism. Then its image is irreducible and complete. By the example above its not $\mathbb{A}^1$. Hence it is a point and f is constant. This shows $k[X] = k$ so X is a point.)

We also need the following:

Lemma 6.11. *Let G be an algebraic group acting transitively on varieties X, Y. Let $\phi : X \to Y$ be a bijective, G-equivariant morphism. If Y is complete, then X is complete.*

Proof. We know that $\phi \otimes \mathrm{id} : X \times Z \to Y \times Z$ is open by 4.6. Since it is bijective too, we deduce that $\phi \otimes \mathrm{id}$ is closed. Now, $pr_Z : X \times Z \to Z$ factors through $X \times Z \to Y \times Z \to Z$ where both are closed on the way. Hence X is complete. $\square$

Now we can prove the all important *Borel's fixed point theorem*:

Theorem 6.12. *Let G be a connected solvable algebraic group, and X be a non-empty complete G variety. Then, G has a fixed point on X.*

Proof. Proceed by induction on $\dim G$, the case $G = \{1\}$ being trivial. Suppose then that $\dim G > 0$ and let $H = G'$, which is connected solvable of strictly smaller dimension. By induction,

$$Y = \{x \in X \mid Hx = x\}$$

is non-empty. It is closed, hence complete, and G stabilizes Y as $H \lhd G$. So we may as well replace X by Y to assume that $H \subseteq G_x$ for all $x \in X$. Since G/H is abelian, this implies that each $G_x \trianglelefteq G$.

Now choose x so that $G.x$ is of minimal dimension. Then, $G.x$ is closed hence complete. The map $G/G_x \to G.x$ is bijective, so we deduce that G/G_x is complete by the preceeding lemma. But G/G_x is affine as $G_x \trianglelefteq G$. So in fact G/G_x is a point, i.e. $G = G_x$ and x is a fixed point. $\square$

Corollary 6.13 (Lie-Kolchin theorem)**.** *Let G be a connected solvable subgroup of $GL(V)$. Then G fixes a flag in V.*

Proof. Let G act on the flag variety $\mathscr{F}(V)$. This is projective, so G has a fixed point. $\square$