## 5. Root systems and semisimple Lie algebras

5.1. **Characteristic 0 theory.** Assume in this subsection that $chark = 0$. Let me recall a couple of definitions made earlier: $G$ is called *reductive* if it is connected and has no non-trivial closed connected unipotent normal subgroup, $G$ is called *semisimple* if it is connected and has no non-trivial closed connected solvable normal subgroup. Semisimple groups are reductive, but reductive groups are a little more general (including e.g. the groups $GL_n$ and $D_n$). The reductive algebraic groups are the ones for which there is a beautiful structure theory. In this section I will discuss the situation for semisimple groups over fields of characteristic 0...

Observe right away that a connected $G$ is semisimple if and only if it has no non-trivial closed connected abelian normal subgroup. Analogously, a finite dimensional Lie algebra $\mathfrak{g}$ is called semisimple if it has no non-zero commutative ideal. The main theorem I want to prove is:

**Theorem 5.1.** *Assume $chark = 0$. Then a connected group $G$ is semisimple if and only if $\mathfrak{g}$ is semisimple. In that case,* $\operatorname{Ad} G = G/Z(G) = (\operatorname{Aut}\mathfrak{g})^\circ$*,*

Note for semisimple $G$ that $Z(G)$ is finite. So the theorem almost classifies the semisimple algebraic groups in characteristic 0: the isomorphism type of $G/Z(G)$ at least is classified by the isomorphism type of $\mathfrak{g}$. Since the latter are classified by Dynkin diagrams, so are the centreless semisimple groups.

Recall that if $\mathfrak{g}$ is simple then $G$ is simple over an arbitrary field. But $SL_n$ in characteristic dividing $n$ gave us an example already where $G$ was simple but $\mathfrak{g}$ was not. So the Theorem is completely wrong in positive characteristic...

For the proof, we begin with:

**Lemma 5.2.** *Assume $chark = 0$ and $G$ is a connected algebraic group.*

1. *If $\phi : G \to H$ is a morphism of algebraic groups, then $\ker d\phi = L(\ker\phi)$.*
2. *If $A, B$ are closed subgroups of $G$ than $\mathfrak{a} \cap \mathfrak{b} = L(A \cap B)$.*

*Proof.* (1) WLOG $\phi$ is surjective. It is automatically separable, so we can identify $H = G/\ker\phi$ by the first isomorphism theorem. But $L(G/\ker\phi) = \mathfrak{g}/L(\ker\phi)$. Therefore $\ker d\phi = L(\ker\phi)$.

(2) Let $\pi : G \to G/B$ be the canonical map, so $\ker d\pi = \mathfrak{b}$. Let $\pi' : A \to G/B$ be the restriction. The fibres of $\pi'$ are the cosets $x(A \cap B)$ for $x \in A$, and $\pi'$ is separable automatically. Therefore we can identify $\pi'$ with the canonical map $A \to A/A \cap B$, hence $\ker d\pi' = L(A \cap B)$ by (1). But clearly $\ker d\pi' = \mathfrak{a} \cap \ker d\pi = \mathfrak{a} \cap \mathfrak{b}$. □

**Lemma 5.3.** *Assume $chark = 0$. Let $\phi : G \to GL(V)$ be a finite dimensional representation. Then, $G$ and $\mathfrak{g}$ leave invariant the same subspaces (resp. vectors) of $V$.*

*Proof.* We may assume $G < GL(V)$. Consider a subspace $W$ of $V$. Let $stab_{GL(V)}(W) = \{g \in GL(V) | gW \subseteq W\}, stab_{\mathfrak{gl}(V)}(W) = \{X \in \mathfrak{gl}(V) | XW \subseteq W\}$.

Using dimensions, you check easily that $L(stab_{GL(V)}(W)) = stab_{\mathfrak{gl}(V)}(W)$ (in arbitrary characteristic!). But

$$stab_G(W) = G \cap stab_{GL(V)}(W), stab_{\mathfrak{g}}(W) = \mathfrak{g} \cap stab_{\mathfrak{gl}(V)}(W).$$

Now using the lemma, you get that $L(stab_G(W)) = stab_{\mathfrak{g}}(W)$. Finally, $G$ stabilizes $W$ if and only if the left hand side is $G$, while $\mathfrak{g}$ stabilizes $W$ if and only if the right hand side is $\mathfrak{g}$. The statement about subspaces follows. The argument for vectors is similar. □

Let $G$ be a connected algebraic group. The subalgebras of $\mathfrak{g}$ that are of the form $L(H)$ for a closed connected subgroup $H$ of $G$ are called the *algebraic subalgebras* of $\mathfrak{g}$. Note (even in characteristic 0) $\mathfrak{g}$ may have subalgebras that are not algebraic.

**Theorem 5.4.** *("Lattice correspondence") Assume* $\operatorname{char} k = 0$ *and* $G$ *is connected. Then the map* $H \mapsto \mathfrak{h}$ *is a 1–1, inclusion preserving correspondence between the closed connected subgroups of* $G$ *and the algebraic subalgebras of* $\mathfrak{g}$*. Moreover, normal subgroups correspond to ideals under the correspondence.*

*Proof.* Suppose that $L(H) = L(K)$. We need to show that $H = K$ (which is *false* in positive characteristic!). But $L(H \cap K) = L(H) \cap L(K)$ by the lemma. Hence $L(H \cap K) = L(H)$, hence $\dim H \cap K = \dim H$ so $H \cap K = H$. Similarly $H \cap K = K$.

Now consider the final statement about normal subgroups. We need to show $H$ is normal in $G$ if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. We already proved the forward implication in chapter 3, using $d\,\mathrm{Int} = \mathrm{Ad}, d\,\mathrm{Ad} = \mathrm{ad}$. Conversely, suppose $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Then, $\mathfrak{g}$ stabilizes $\mathfrak{h}$ acting via $\mathrm{ad} = d\,\mathrm{Ad}$. So by the Lemma, $G$ stabilizes $\mathfrak{h}$ acting via Ad. So for $x \in G$, $\mathrm{Ad}\,x(\mathfrak{h}) = \mathfrak{h}$. But $\mathrm{Ad}\,x : \mathfrak{h} \to \mathfrak{g}$ is the differential of $\mathrm{Int}\,x : H \to G$. So the image $\mathfrak{h} = (\mathrm{Ad}\,x)(\mathfrak{h}) = L(\mathrm{Int}\,x(H)) = L(xHx^{-1})$ by separability. Now by the previous paragraph, we get that $H = xHx^{-1}$ because they have the same Lie algebra, hence $H$ is normal in $G$. □

Note this already shows $G$ is simple if and only if $\mathfrak{g}$ is simple. To prove the main theorem above, we need one more lemma.

**Lemma 5.5.** *Let* $\operatorname{char} k = 0$ *and* $G$ *be a connected algebraic group. For* $x \in G$, $L(C_G(x)^0) = \mathfrak{c}_{\mathfrak{g}}(x)$.

*Proof.* Recall $C_G(x) = \{g \in G \mid xgx^{-1} = g\}$ and $\mathfrak{c}_{\mathfrak{g}}(x) = \{X \in \mathfrak{g} \mid \mathrm{Ad}\,xX = X\}$. In general, it is obvious that

$$L(C_G(x)^0) \subseteq \mathfrak{c}_{\mathfrak{g}}(x).$$

For $\mathrm{Int}\,x : C_G(x)^0 \to C_G(x)^0$ is the identity map, so its differential $\mathrm{Ad}\,x$ is the identity map on $L(C_G(x)^0)$.

Now suppose for a moment that $G = GL_n$. Then $C_G(x)$ is all invertible matrices that commute with $x$ and $\mathfrak{c}_{\mathfrak{g}}(x)$ is all matrices that commute with

$x$. Clearly the latter is a principle open subset of the former, so we get equality in this case by dimension.

In general, we may assume $G$ is a closed subgroup of $GL_n$. Then,

$$\mathfrak{c}_{\mathfrak{g}}(x) = \mathfrak{c}_{\mathfrak{gl}_n}(x) \cap \mathfrak{g} = L(C_{GL_n}(x)) \cap L(G) = L(C_{GL_n}(x) \cap G) = L(C_G(x)).$$

Note we have used $char k = 0$ in this last step! □

**Lemma 5.6.** *Let $char k = 0$ and $G$ be a connected algebraic group. Then,* $\ker \mathrm{Ad} = Z(G)$.

*Proof.* We already know $Z(G) \subseteq \ker \mathrm{Ad}$. Since

$$L(C_G(x)) = \mathfrak{c}_{\mathfrak{g}}(x)$$

we see that if $\mathrm{Ad}\, x = 1$ then $L(C_G(x)) = \mathfrak{g}$ so $C_G(x) = G$ by the lattice correspondence, i.e. $x \in Z(G)$. Thus $Z(G) = \ker \mathrm{Ad}$. □

Now we can prove the theorem stated above:

**Theorem 5.7.** *Assume $char k = 0$. Then a connected group $G$ is semisimple if and only if $\mathfrak{g}$ is semisimple. In that case,* $\mathrm{Ad}\, G = G/Z(G) = (\mathrm{Aut}\,\mathfrak{g})^\circ$,

*Proof.* Suppose $\mathfrak{g}$ is semisimple. If $N$ is a closed connected abelian normal subgroup of $G$, then $L(N)$ is an abelian ideal of $\mathfrak{g}$, hence $L(N) = 0$ hence $N = 1$. This shows $G$ is semisimple.

Conversely, suppose $G$ is semisimple. Let $\mathfrak{n}$ be an abelian ideal of $\mathfrak{g}$. Let $H = C_G(\mathfrak{n})^0$, so $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{n})$ by the previous lemma. Moreover, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ (just check $[[X, Y], Z] = 0$ for all $X \in \mathfrak{g}, Y \in \mathfrak{h}, Z \in \mathfrak{n}$ using the Jacobi identity) containing $\mathfrak{n}$ in its center.

By the lattice correspondence, $H$ is a normal subgroup of $G$, so $Z := Z(H)^0$ is also normal in $G$, hence zero as $G$ is semisimple. By 5.5, $Z(H) = \ker \mathrm{Ad}$ so its Lie algebra is $\ker \mathrm{ad} = \mathfrak{z}(\mathfrak{h})$, which contains $\mathfrak{n}$. Hence $\mathfrak{n} = 0$.

It remains to prove that $\mathrm{Ad}\, G = (\mathrm{Aut}\,\mathfrak{g})^0$. I'm not going to do this. □

5.2. **Root systems.** Hopefully the theorem just proved motivates spending a little time looking at the finite dimensional semisimple Lie algebras over $\mathbb{C}$. I want to explain their classification. The first step is to introduce the abstract notion of a *root system.* Here is the definition: a root system is a pair $(E, \Phi)$ where $E$ is a Euclidean space and $\Phi$ is a finite set of vectors, called *roots*, in $E$ such that

- (R1) $0 \notin \Phi$ spans $E$.
- (R2) $\alpha, c\alpha \in \Phi$ implies $c = \pm 1$.
- (R3) $\Phi$ is invariant under the reflection $s_\alpha$ in the hyperplane orthogonal to $\alpha$ (i.e. the automorphism $\beta \mapsto \beta - (\beta, \alpha^\vee)\alpha$ where $\alpha^\vee := 2\alpha/(\alpha, \alpha)$).
- (R4) $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \mathbb{Z}$.

Given a root system, the *Weyl group* $W$ is the subgroup of $GL(E)$ generated by the $s_\alpha$ for $\alpha \in \Phi$. It is a finite group, since it acts faithfully on the finite set $\Phi$. I will give examples in class to give you a geometric way to think about $W$.

We let $H_\alpha = \{\beta \in E \,|\, (\alpha, \beta) = 0\}$ be the hyperplane orthogonal to $\alpha$. The connected components of

$$E - \bigcup_{\alpha \in \Phi} H_\alpha$$

are called the *Weyl chambers*. Fix a chamber $C$, which we will call the *fundamental chamber*. Then one can show that the map

$$w \mapsto wC$$

is a bijection between $W$ and the set of chambers.

The choice of $C$ fixes several other things... We let $\Phi^+$ be the set of all $\alpha \in \Phi$ which are in the same half space as $C$. Then, $\Phi = \Phi^+ \sqcup (-\Phi^+)$. Elements of $\Phi^+$ are called *positive roots*. Next, let

$$\Pi = \{\alpha \in \Phi^+ \mid H_\alpha \text{ is one of the walls of } C\}.$$

This is called a *base* for the root system. One can show that $\Pi$ is actually a basis for the vector space $E$, and moreover every element of $\Phi^+$ is a non-negative integer linear combination of $\Pi$. Elements of $\Pi$ are called *simple roots*.

The Weyl group $W$ is actually generated by the $s_\alpha$ for $\alpha \in \Pi$, i.e. by the reflections in the walls of the fundamental chamber. This leads to the idea of the *length* $\ell(w)$ of $w \in W$. This is defined as the minimal length of an expression $w = s_{\alpha_1} \dots s_{\alpha_r}$ where $\alpha_1, \dots, \alpha_r$ are simple roots. Geometrically, $\ell(w)$ is the number of hyperplanes separating $wC$ from $C$.

Now let $(E, \Phi)$ be a root system. Let $\Pi = \{\alpha_1, \alpha, \alpha_\ell\}$ be a base coming from a choice of fundamental chamber. Here $\ell = \dim E$ is the *rank* of the root system. The *Cartan matrix* $A = (a_{i,j})_{1 \leq i,j \leq \ell}$ is the matrix with

$$a_{i,j} = (\alpha_i, \alpha_j^\vee).$$

Since all the Weyl chambers are conjugate under the action of $W$, the Cartan matrix is an invariant of the root system (up to simultaneous permutation of rows/columns). Here are some basic properties about this matrix:

(C1) $a_{i,i} = 2$.
(C2) For $i \neq j$, $a_{i,j} \in \{0, -1, -2, -3\}$.
(C3) $a_{i,j} \neq 0$ if and only if $a_{j,i} \neq 0$.

Note (C2) is not obvious. It follows because $E' = \langle \alpha_i, \alpha_j \rangle$ together with $\Phi' := \Phi \cap E'$ is a root system of rank 2. The Cartan matrices for the rank two root systems are exactly the following:

$$A_1 \times A_1 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, B_2 : \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Note if $a_{i,j} \neq 0$, then

$$(\alpha_i, \alpha_i)/(\alpha_j, \alpha_j) = a_{i,j}/a_{j,i} \in \{1, 2, 3\}.$$

So you can work out the ratio of the lengths of the roots $\alpha_i, \alpha_j$ to each other from the Cartan matrix. A root system is called *indecomposable* if it cannot be partitioned $E = E_1 \perp E_2, \Phi = \Phi_1 \sqcup \Phi_2$ where $(E_i, \Phi_i)$ are root systems. For an indecomposable root system, you can work out the ratio of lengths of any pair of roots to each other from the Cartan matrix, hence you completely recover the form $(.,.)$ on $E$ *up to a scalar* from the Cartan matrix. You also recover $\Phi$, since the Cartan matrix contains enough information to compute the reflection $s_{\alpha_i}$ for each $i = 1, \dots, \ell$, and $\Phi = W\Pi$. So: an indecomposable root system is completely determined up to isomorphism by its Cartan matrix.

A convenient shorthand for Cartan matrices is given by the *Dynkin diagram.* This is a graph with vertices labelled by $\alpha_1, \dots, \alpha_\ell$. There are $a_{i,j}a_{j,i}$ edges joining vertices $\alpha_i$ and $\alpha_j$, with an arrow pointing towards $\alpha_i$ if $(\alpha_i, \alpha_i) < (\alpha_j, \alpha_j)$ (equivalently, $a_{i,j} = -1, a_{j,i} = -2, -3$). Clearly you can recover the Cartan matrix from the Dynkin diagram given properties (C1)–(C3) above.

Now I can state the classification of root systems:

**Theorem 5.8.** *The Dynkin diagrams of the indecomposable root systems are (...).*

**Exercise 5.9.** (5) In class I wrote down the explicit construction of the root system of type $A_\ell$, and showed that the length of the longest element $w_0$ of the Weyl group was $\ell(\ell+1)/2$. Do the same thing for the other classical root systems $B_\ell, C_\ell$ and $D_\ell$. (You will need to look them up!! There are many good sources, e.g. Humphreys' "Introduction to Lie algebras and representation theory", Bourbaki "Groupes et Algebres de Lie", Kac "Infinite dimensional Lie algebras", Carter "Finite groups of Lie type", Helgason "Differential geometry and symmetric spaces"...)

5.3. **Semisimple Lie algebras.** Now I sketch how the semisimple Lie algebras are classified by the root systems. Of course, we need to start with a semisimple Lie algebra and build a root system out of it, and vice versa.

Let us begin with a *finite dimensional semisimple Lie algebra* $\mathfrak{g}$ over $\mathbb{C}$. Here are some equivalent statements:

(1) $\mathfrak{g}$ is semisimple;
(2) $\mathfrak{g}$ has no proper abelian ideals;
(3) $\mathfrak{g}$ possesses a *non-degenerate invariant* symmetric bilinear form $(.,.)$, where invariant here means $([X,Y],Z) = (X,[Y,Z])$. Note if $\mathfrak{g}$ is simple, there is a unique such form up to a scalar...

Proving the equivalence of (1) and (2) (not at all obvious) is the first major step in developing the theory. The experts will know that there is a "canonical" choice of non-degenerate form, the Killing form, but we don't need that here.

**Example 5.10.** Let us consider $\mathfrak{sl}_n$. The bilinear form $(X,Y) = \mathrm{tr}(XY)$ is non-degenerate and invariant. So $\mathfrak{sl}_n$ is a semisimple Lie algebra. Let $e_{i,j}$ be the $ij$-matrix unit and let $\mathfrak{h}$ be the diagonal, trace zero matrices. We can decompose

$$\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C} e_{i,j}.$$

A basis for $\mathfrak{h}$ is given by $h_1, \dots, h_{n-1}$ where $h_i = e_{i,i} - e_{i+1,i+1}$. Let $\epsilon_i \in \mathfrak{h}^*$ be the map sending a diagonal matrix to its $i$th diagonal entry. Note $\epsilon_1 + \cdots + \epsilon_n = 0$, i.e. the $\epsilon_i$'s are not independent. Then:

$$[H, e_{i,j}] = (\epsilon_i - \epsilon_j)(H) e_{i,j},$$

i.e. $e_{i,j}$ is a simultaneous eigenvector for $\mathfrak{h}$. We use the word *weight* in place of eigenvalue, so $e_{i,j}$ is a vector of *weight* $\epsilon_i - \epsilon_j$. Now you recall that the root system of type $A_{n-1}$ can be defined as the real vector subspace of $\mathfrak{h}^*$ spanned by $\epsilon_1, \dots, \epsilon_n$, and the roots are

$$\Phi := \{\epsilon_i - \epsilon_j \mid i \neq j\}.$$

A base for $\Phi$ is given by by $\alpha_1, \dots, \alpha_{n-1}$ where $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Let us finally write $\mathfrak{g}_\alpha := \mathbb{C} e_{i,j}$ if $\alpha = \epsilon_i - \epsilon_j$, i.e. the weight space of $\mathfrak{g}$ of weight $\epsilon_i - \epsilon_j$. Then:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

In other words, you "see" the root system of type $A_{n-1}$ when you decompose $\mathfrak{g}$ into weight spaces with respect to the diagonal matrices. Final note: the inner product giving the Euclidean space structure is induced by the non-degenerate form defined to start with... Indeed if you compute the matrix $(h_i, h_j)$ you get back the Cartan matrix of type $A_{n-1}$...

This example is more or less how things go in general, when you start with an arbitrary semisimple Lie algebra $\mathfrak{g}$, with a non-degenerate invariant form $(.,.)$. The first step is to develop in $\mathfrak{g}$ a theory of Jordan decompositions. This parallels the Jordan decomposition we proved for algebraic groups! You call an element $X$ of $\mathfrak{g}$ *semisimple* if the linear map $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable, and *nilpotent* if $\operatorname{ad} X$ is nilpotent. The *abstract Jordan decomposition* shows that any $X \in \mathfrak{g}$ decomposes uniquely as $X = X_s + X_n$ where $X_s \in \mathfrak{g}$ is semisimple and $X_n \in \mathfrak{g}$ is nilpotent, and $[X_s, X_n] = 0$.

What is more, if you have a *representation of* $\mathfrak{g}$, i.e. a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}_n$, it is true that $\rho(X_s) = \rho(X)_s$ and $\rho(X_n) = \rho(X)_n$, where the semisimple and nilpotent parts on the right hand side are taken just as $n \times n$ matrices in $\mathfrak{gl}_n$. Thus, the abstract Jordan decomposition is consistent with all other Jordan decompositions arising from all other representations. In particular, semisimple elements of $\mathfrak{g}$ map to diagonalizable matrices under any matrix representation of $\mathfrak{g}$. For $\mathfrak{sl}_n$, $e_{i,j}$ is nilpotent for $i \neq j$ and semisimple for $i = j$.

Now you introduce the notion of a *maximal toral subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$. This is a maximal abelian subalgebra all of whose elements are semisimple. It turns out that in a semisimple Lie algebra, maximal toral subalgebras are non-zero, and they are all conjugate under automorphisms of $\mathfrak{g}$. Now fix one – it doesn't really matter which, since they are all conjugate. Importantly, the restriction of the invariant form $(.,.)$ on $\mathfrak{g}$ to $\mathfrak{h}$ is still non-degenerate. So we can define a map

$$\mathfrak{h}^* \to \mathfrak{h}$$

mapping $\alpha \in \mathfrak{h}^*$ to $t_\alpha \in \mathfrak{h}$, where $t_\alpha$ is the unique element satisfying $(t_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. Now we can even lift the non-degenerate form on $\mathfrak{h}$ to $\mathfrak{h}^*$, by defining $(\alpha, \beta) = (t_\alpha, t_\beta)$. Thus, $\mathfrak{h}^*$ now has a non-degenerate symmetric bilinear form on it too.

For $\alpha \in \mathfrak{h}^*$, define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{g}\}.$$

Clearly, $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$. Set $\Phi = \{0 \neq \alpha \in \mathfrak{h}^* \,|\, \mathfrak{g}_\alpha \neq 0\}$. Then you get *Cartan decomposition of* $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

(it is not quite obvious that the right hand side is everything...). It turns out with some work that each of the $\mathfrak{g}_\alpha$ spaces are *one dimensional*.

Now you can build a root system out of $\mathfrak{g}$: we've already constructed the set $\Phi$. Let $E$ be the real vector subspace of $\mathfrak{h}^*$ spanned by $\Phi$. The restriction of the form on $\mathfrak{h}^*$ to $E$ turns out to be real valued only, and makes $E$ into a Euclidean space. Now:

**Theorem 5.11.** *The pair $(E, \Phi)$ just built out of $\mathfrak{g}$ (starting from a choice of $\mathfrak{h}$) is a root system. Moreover, the resulting map from semisimple Lie algebras to Dynkin diagrams gives a bijection between isomorphism classes of semisimple Lie algebras and Dynkin diagrams. The decomposition of a semisimple Lie algebra as a direct sum of simples corresponds to the decomposition of the Dynkin diagram into connected components.*

For example, $\mathfrak{sl}_n$ is the *simple* Lie algebra corresponding to the Dynkin diagram $A_{n-1}$. One other point. When you study root systems, you fix a choice of fundamental chamber, giving you the notion $\Phi^+$ of positive roots and $\Pi$ of simple roots. This fixes in $\mathfrak{g}$ a *triangular decomposition*... Borel subalgebras...

**Exercise 5.12.** (6) Look up or work out the dimensions of the simple Lie algebras of types $A_\ell, B_\ell, C_\ell$ and $D_\ell$. In particular, check that $\dim C_\ell$ is the same as the dimension of the algebraic group $Sp_{2\ell}$ from exercise (2).