4. Quotients

4.1. **More algebraic geometry.** We will always be concerned with a dominant morphism $\phi : X \to Y$ between irreducible varieties. Recall that (as ϕ is dominant) we have the comorphism

$$\phi^* : k(Y) \hookrightarrow k(X)$$

which allows us to view $k(X)$ as a field extension of $k(Y)$. If ϕ^* is an *isomorphism* then the morphism ϕ is said to be *birational*.

Lemma 4.1. *ϕ is birational if and only if there is a non-empty open subset U of X such that ϕU is open in Y and $\phi : U \to \phi U$ is an isomorphism.*

Proof. ($\Leftarrow$). This immediately implies that ϕ is birational by the definition of function fields.

($\Rightarrow$). Assume ϕ is birational. Without loss of generality, we can take X and Y to be affine. Then, $\phi^* : k[Y] \hookrightarrow k[X]$ as ϕ is dominant, and the induced map $\phi^* : k(Y) \to k(X)$ at the level of fields of fractions is an isomorphism. So $k[Y] \subseteq k[X] \subseteq k(Y)$ and we can write $k[X] = k[Y][f_1, \ldots, f_r]$ for $f_1, \ldots, f_r \in k(Y)$. Putting the f_i over a common denominator, we get $0 \neq f \in k[Y]$ such that all $f_i f \in k[Y]$. Let $U = D_X(f)$, a principal open subset of X. Then, $\phi(U) = D_Y(f)$ and $\phi^* : k[Y]_f \to k[X]_f$ is an isomorphism, hence $U \cong \phi(U)$. □

Two irreducible varieties are said to be *birationally equivalent* if there exists a birational morphism between them. This is an important notion in algebraic geometry: birational equivalence is a much weaker thing than being isomorphic, and it turns out to be much more reasonable to classify varieties up to birational equivalence rather than up to isomorphism.

Example 4.2. Let $Y = \{(x, y) \in \mathbb{A}^2 | x^2 = y^3\}$ be the twisted cubic. Define a morphism $\phi : \mathbb{A}^1 \to Y$ by $\phi(x) = (x^3, x^2)$. This is obviously a bijective map. The inverse is $(x, y) \mapsto x/y$ for $(x, y) \neq (0, 0)$ and $(0, 0) \mapsto 0$. However, the inverse map is *not* a morphism of varieties (what would its comorphism be?). So it is not an isomorphism of varieties. But ϕ is birational: consider $\mathbb{A}^1 - \{0\}$ which ϕ maps to $Y - \{(0, 0)\}$. This is an isomorphism of varieties: the inverse map is $(x, y) \mapsto x/y$ which IS a morphism of varieties on $Y - \{(0, 0)\}$ – you can invert y on this open subset.

We've just given an example of a birational, bijective morphism that is *not* an isomorphism. You might think the problem here is Y: it is not a smooth variety since $(0, 0)$ is a cusp. You would be right:

Theorem 4.3. (Zariski's main theorem) *Let $\phi : X \to Y$ be a morphism of irreducible varieties that is bijective and birational. Assume that Y is a smooth variety. Then, ϕ is an isomorphism.*

Remark 4.4. I am not going to prove the theorem. We will apply it at a crucial point in the next subsection in our construction of the quotient of an algebraic group by a closed subgroup. The real statement of Zariski's main theorem replaces the word "smooth" with the word "normal". I do not want to get into the definition and properties of normal varieties, so I have stated the weaker result above, which will be enough for our purposes.

Suppose that G is a connected algebraic group acting on a variety X. Let $x \in X$. I have already told you that

$$\dim Gx = \dim G - \dim G_x.$$

The next result I want to state proves this important fact, actually it is a much stronger fact about arbitrary morphisms of varieties.

Theorem 4.5. *Let X and Y be irreducible varieties and $\phi : X \to Y$ be a dominant morphism. Let $r = \dim X - \dim Y \geq 0$. Then, there is a non-empty open subset U of X with the following properties:*

(i) *For any variety Z the restriction of $\phi \times 1$ to $U \times Z$ defines an open morphism $U \times Z \to Y \times Z$.*
(ii) *If Y' is an irreducible closed subvariety of Y and X' is an irreducible component of $\phi^{-1}Y'$ intersecting U, then $\dim X' = \dim Y' + r$. In particular, if $y \in Y$, then any irreducible component of $\phi^{-1}y$ intersecting U has dimension r.*
(iii) *If $r = 0$, then for all $x \in U$, the number of points in the fibre $\phi^{-1}(\phi x)$ is equal to the degree of the field extension $k(X)_s/K(Y)$, where $k(X)_s$ is the set of all elements of $k(X)$ that are separable over $k(Y)$.*

Roughly speaking, the theorem says that a morphism is always well-behaved *in some open subset*. It can be quite badly behaved outside of that open set. As usual when there is a transitive action of an algebraic group involved, we can get a stronger result as a corollary:

Corollary 4.6. *Let G be a connected algebraic group and X, Y be varieties on which G acts transitively. Suppose $\phi : X \to Y$ is a G-equivariant morphism. Let $r = \dim X - \dim Y$. Then:*

(i) *For any variety Z, the morphism $\phi \times 1 : X \times Z \to Y \times Z$ is open.*
(ii) *If Y' is an irreducible closed subvariety of Y and X' an irreducible component of $\phi^{-1}Y'$ then $\dim X' = \dim Y' + r$. In particular, for $y \in Y$, all irreducible components of $\phi^{-1}y$ have dimension r.*
(iii) *ϕ is an isomorphism if and only if it is bijective and for some $x \in X$ the differential $d\phi_x : T_x X \to T_{\phi x} Y$ is onto.*

Proof. (i), (ii) Let $U \subseteq X$ be an open subset as in the theorem. Then all translates gU enjoy the the same properties. Since these cover X we get (i) and (ii).

(iii) Suppose that ϕ is bijective and $d\phi_x$ is onto. Then, by (ii), $r = 0$ and by Theorem 4.5(iii), we must have that $k(X)_s = k(Y)$, i.e. $k(X)$ is a purely inseparable extension of $k(Y)$. On the other hand, by the differential criterion for separability 3.16(ii), $d\phi_x$ onto implies that ϕ is a separable morphism, i.e. $k(X)$ is a separable extension of $k(Y)$. Hence, $k(X) = k(Y)$ and ϕ is birational. Finally, since Y is an orbit, it is smooth, so we get that ϕ is an isomorphism by Zariski's main theorem. □

Corollary 4.7. *Let $\phi : G \to H$ be a homomorphism of algebraic groups. Then:*

(i) $\dim G = \dim \operatorname{im} \phi + \dim \ker \phi$.
(ii) *ϕ is an isomorphism if and only if it is bijective and the differential $d\phi_e$ is onto.*

Proof. This follows easily from Corollary 4.6. □

Corollary 4.8. *Suppose G is a connected algebraic group acting on a variety X. Then, for $x \in X$, $\dim Gx = \dim G - \dim G_x$.*

Proof. Let $\phi : G \to Gx, g \mapsto gx$ be the orbit map. By Corollary 4.6(ii), $\dim \phi^{-1}x = \dim G - \dim Gx$. But $\phi^{-1}x$ is the stabilizer G_x. □

I am not planning on proving Theorem 4.5, rather I want to give some examples to give you the flavor of how things can go wrong in general – hopefully that is more useful!

Example 4.9. (1) Consider the morphism $\phi : \mathbb{A}^2 \to \mathbb{A}^2, (x, y) \mapsto (x, xy)$. I first want to show that this is a birational morphism that is not an open morphism (so the subset U given by the theorem must be a *proper* subset in this example).

Let $U = \{(x, y) \in \mathbb{A}^2 \mid x \neq 0\}$. Clearly, $\phi(U) = U$. Moreover, on U we can invert ϕ: the inverse map sends $(x, z) \mapsto (x, z/x)$ – this is a morphism because we have localized to invert x. Therefore $\phi|_U : U \to U$ is an isomorphism of varieties, and ϕ is birational.

Now consider the image of ϕ on all of $\mathbb{A}^2$. It is $\mathbb{A}^2 - \{(0, y) \mid y \neq 0\}$. That is not an open subset of $\mathbb{A}^2$. Hence, ϕ itself is not an open morphism.

Finally, let's compute the irreducible components of the fibres of ϕ: certainly for $x \in U$, the fibres are just points. So consider $x \in \phi(\mathbb{A}^2) - U$. There is only one x to consider: $x = (0, 0)$. Then,

$$\phi^{-1}(0, 0) = \{(0, y) \mid y \in k\}$$

which is one dimensional. Thus the fibres of ϕ are points in the open set U, and outside of that they blow up.

(2) For another example, consider $\phi : \mathbb{A}^3 \to \mathbb{A}^3, (x, y, z) \mapsto (x, xy, z)$. Let $X = \{(x, y, z) \in \mathbb{A}^3 \mid y^2 = 1 + x\}$. Let $Y = \phi(X)$. I first want to show that X and Y are irreducible, closed of dimension 2.

For X, we have that

$$k[X] = k[T_1, T_2, T_3]/(T_2^2 - T_1 - 1).$$

I claim that $k[X] \cong k[S_1, S_2]$. The proof uses the fact that the polynomial rings are free: the map $T_1 \mapsto S_2^2 + 1, T_2 \mapsto S_2, T_3 \mapsto S_3$

factors through $k[X]$. Conversely, we can map $S_2 \mapsto \overline{T_2}, S_3 \mapsto \overline{T_3}$ to get the inverse. This gives at once that X is irreducible and closed of dimension 2.

Consider Y. We have that

$$Y = \{(x, w, z) \mid w^2 = x^2 + x^3\}.$$

So it is closed, and it is irreducible since its the image of an irreducible under a morphism. Let $U = \{(x, y, z) \in X \mid x \neq 0\}$, which is open in X. Then, $\phi U = \{(x, w, z) \in Y \mid x \neq 0\}$, so is open in Y. The morphism $\phi|_U : U \to \phi U$ is an isomorphism: the inverse maps (x, w, z) to $(x, w/x, z)$. Thus, ϕ is birational, hence $\dim Y = \dim X = 2$ too.

By the theorem, we expect for a closed subvariety Y' of Y that the "generic" irreducible components of $\phi^{-1}Y'$ have the same dimension as Y'. Consider

$$Y' = \{(x, y, z) \in Y \mid y = xz, z^2 = 1 + x\}.$$

I claim that Y' is irreducible of dimension 1. To see this, note

$$k[Y'] = k[T_1, T_2, T_3]/(T_2^2 - T_1^2 - T_1^3, T_2 - T_1T_3, T_3^2 - T_1 - 1).$$

I claim this is isomorphic to $k[S]$, the polynomial ring in one variable. One way, we map $S \mapsto T_3$. The inverse is induced by $T_1 \mapsto S^2 - 1, T_2 \mapsto S^3 - S, T_3 \mapsto S$. One has to check that T_2^2 maps to the same thing as $T_1^2 + T_1^3$, i.e. $(S^3 - S)^2 = (S^2 - 1)^2 + (S^2 - 1)^3$. Hence $Y' \cong \mathbb{A}^1$.

Finally, consider the irreducible components of $\phi^{-1}Y'$. It equals

$$\{(x, y, z) \in X \mid xy = xz, z^2 = 1 + x\}.$$

So either we have $x = 0, z^2 = 1, y^2 = 1$ or $x = z^2 - 1, y = z$. It follows that, providing $char k \neq 2$, the irreducible components of $\phi^{-1}Y'$ are

$$\{(0, 1, -1)\} \cup \{(0, -1, 1)\} \cup \{(z^2 - 1, z, z) \mid z \in k\}$$

The last of this is $\cong \mathbb{A}^1$, and is the "generic case". The first two irreducible components are 0 dimensional so do not meet the "well-behaved" open set U.

4.2. **Chevalley's theorem.** Let G be an algebraic group, H a closed subgroup. We want to give the set $G/H = \{gH \mid g \in G\}$ of cosets the structure of an algebraic variety. I am going to assume for simplicity that G is connected. Everything I am saying here can also be done for non-connected groups G without very much more work...

Write $\mathfrak{g}$ and $\mathfrak{h}$ for their Lie algebras. Recall that $\mathfrak{g} = T_eG$. But we identified this via a map $\sim: T_eG \to L(G)$ with the Lie algebra of all left invariant derivations of $k[G]$. In particular, elements of $\mathfrak{g}$ are endomorphisms of $k[G]$ via $\sim$. Moreover, we have shown that

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \tilde{X} I(H) \subseteq I(H)\},$$

see 3.20.

Lemma 4.10. *Let H be a closed subgroup of G. There exists a finite dimensional subspace V of $k[G]$ and a subspace W of V such that V is stable under all right translations ρ_x for $x \in G$ and*

$$H = \{x \in G \mid \rho_x W = W, \qquad \mathfrak{h} = \{X \in \mathfrak{g} \mid XW \subseteq W\}$$

Proof. Let $I = I(H)$. Let V be a finite dimensional subspace of $k[G]$ stable under all ρ_x's and containing a linearly independent set of generators $f_1, \dots, f_r$ of I (this is possible by 2.4). Let $W = V \cap I$.

Now let us check that the lemma is satisfied by this data. Clearly the ρ_x for $x \in H$ stabilize both V and I, hence W. Suppose $x \in G$ stabilizes W. Then it stabilizes I since W generates I. Therefore for each $f \in I$, we have that $f(x) = (\rho_x f)(e) = 0$ as $\rho_x f \in I$. This shows that $x \in H$.

Finally consider $\mathfrak{h} = \{X \in \mathfrak{g} \mid \tilde{X} I \subseteq I\}$. Let $\rho : G \to GL(V)$ be the right regular representation of G on V. Let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be its differential. Then, $\tilde{X} f = d\rho(X) f$, so all $\tilde{X}$ leave V invariant. Combine these two facts... □

Lemma 4.11. *Let $W < V$ be finite dimensional vector spaces, $d = \dim W$. Let L be the one dimensional subspace $\bigwedge^d W < \bigwedge^d V$. Let $\phi : GL(V) \to GL(\bigwedge^d V)$ be the natural morphism.*

1. *For $x \in GL(V)$, $xW = W$ if and only if $\phi(x) L = L$.*
2. *For $X \in \mathfrak{gl}(V)$, $XW \subseteq W$ if and only if $d\phi(X) L \subseteq L$.*

Proof. Recall $\phi(x)(v_1 \wedge \cdots \wedge v_d) = xv_1 \wedge \cdots \wedge xv_d$. Also, which you can check yourself, $d\phi(X)(v_1 \wedge \cdots \wedge v_d) = \sum_{i=1}^d v_1 \wedge \cdots \wedge Xv_i \cdots \wedge v_d$. Now you have to do some linear algebra. For instance, for (i), pick a basis $v_1, \dots, v_d$ for W and extend to a basis $v_1, \dots, v_n$ for V. We can to this in such a way that $v_{l+1} \wedge \cdots \wedge v_{l+d}$ os a basis of xW, for some $l \geq 0$. Then $v_1 \wedge \cdots \wedge v_d$ is a basis for L and $v_{l+1} \wedge \cdots \wedge v_{l+d}$ is a basis of $\phi(x)L$. If $l > 0$ then these are linearly independent so that x does not stabilize L. Hence we must have that $l = 0$. □

Combining the lemmas gives at once:

Theorem 4.12. *Let H be a closed subgroup of G. There exists a representation $\phi : G \to GL(V)$ and a non-zero $v \in V$ such that*

$$H = \{x \in G \mid (\phi x) v \in kv\}, \mathfrak{h} = \{X \in \mathfrak{g} \mid (d\phi X) v \in kv\}.$$

Corollary 4.13. *Let H be a closed subgroup of $G = G^0$. There exists a quasi-projective variety X that G acts transitively on and a point $x \in X$ such that*

(i) $G_x = H$.
(ii) *The orbit map* $\psi : G \to X, g \mapsto gx$ *is separable.*
(iii) *the fibres of* ψ *are the cosets* gH *for* $g \in G$.

Proof. Let V and $v \in V$ be as in the theorem. Take X to be the G-orbit $G\langle v \rangle$ in $\mathbb{P}(V)$. This is open in its closure, hence it is a quasi-projective variety. By the Theorem, H is the stablizer of $\langle x \rangle$, hence using transitivity of the action, the fibres of the orbit map are exactly the cosets of H.

Finally to get that the orbit map is separable, note that the tangent space to $\mathbb{P}(V)$ at $\langle x \rangle$ can be canonically identified with $V/\langle x \rangle$. The tangent space to X at $\langle x \rangle$ is therefore a subspace of $V/\langle x \rangle$. The differential $d\psi_e$ maps $X \in \mathfrak{g}$ to $Xx/\langle x \rangle$. So the kernel of the differential is the stabilizer of $\langle x \rangle$, i.e. $\mathfrak{h}$. So,

$$\dim \ker d\psi_e = \dim \mathfrak{h} = \dim H = \dim G - \dim X.$$

Hence $d\psi_e$ is onto by dimension and ψ is separable by 3.16. □

4.3. **Quotients.** Continue with G a connected algebraic group, H a closed subgroup. The assumption G connected can easily be dropped, but it makes things slightly simpler...

Define a *Chevalley quotient* of G by H to be a variety X together with a surjective, separable morphism $\pi : G \to X$ such that the fibres of π are exactly the cosets xH of H in G. We have just shown 4.13 that Chevalley quotients always exist. On the other hand, it is far from clear that they are unique even up to isomorphism!

Define a *categorical quotient* of G by H to be a variety X together with a morphism $\pi : G \to X$ that is constant on all cosets xH of H in G with the following universal property: given any other variety Y and morphism $\phi : G \to Y$ that is constant on all xH's, there is a unique morphism $\bar{\phi} : X \to Y$ such that $\phi = \bar{\phi} \circ \pi$. It is obvious that if a categorical quotient exists, then it is unique up to canonical isomorphism. But it is far from obvious that they should exist at all!

Goal: prove that Chevalley quotients are categorical quotients. (In particular, this will give us that categorical quotients exist, and that Chevalley quotients are unique up to canonical isomorphism.) So we need to take a Chevalley quotient (X, π) and check that it has the right universal property. Given a morphism $\phi : G \to Y$ constant on cosets, there is obviously a unique map of *sets* $X \to Y$ factoring ϕ, since the cosets of π are exactly the cosets. But it is almost impossible from this point of view to prove that this map is a morphism of varieties!!! So we need to proceed rather differently.

Theorem 4.14. *Chevalley quotients are categorical quotients.*

Proof. Step one. Let us try to construct a categorical quotient not in the category of varieties but in the more general category of ringed spaces. Define G/H to be the set gH of cosets of H in G. Let $\pi : G \to G/H$ be the map $g \mapsto gH$. Give G/H the structure of a topological space by declaring

$U \subseteq G/H$ to be open if and only if $\pi^{-1}U$ is open. (This is the coarsest topology on G/H so that π is an open map). Next define a sheaf $\mathcal{O}$ of functions on G/H: if $U \subseteq G/H$ is open, let $\mathcal{O}(U)$ consist of all functions f on U such that $f \circ \pi \in \mathcal{O}_G(\pi^{-1}U)$. (Check the sheaf axioms!)

Now let us check that $(G/H, \pi)$ as just defined is a quotient of G by H in the category of ringed spaces. So let $\psi : G \to Y$ be a morphism of ringed spaces constant on cosets of H in G. We get induced a unique map as sets $\bar{\psi} : G/H \to Y, gH \mapsto \psi(g)$. We need to check it is a morphism of ringed spaces. For continuity, note that for $V \subseteq Y$ open, $U := \bar{\psi}^{-1}V$ is open as $\pi^{-1}(\bar{\psi}^{-1}V) = \psi^{-1}V$ is open in G. Now we show that $\bar{\psi}^*$ maps $f \in \mathcal{O}_Y(V)$ to an element of $\mathcal{O}_{G/H}(U)$. By definition, we just need to check that $\pi^*(\bar{\psi}^* f) \in \mathcal{O}_G(\psi^{-1}V)$. But that is $\psi^* f$ which does since ψ was a morphism of ringed spaces to start with. We are done.

Okay, so now we have constructed $(G/H, \pi)$, a quotient of G by H in the category of ringed spaces. It was quite easy, but *there is absolutely no guarantee that G/H should be a variety!!* (If it is, then it is also a quotient in the category of varieties and we are done).

Step two. Now let $(G/H, \pi)$ be as in step one, and let (X, ψ) be a Chevalley quotient. Using the universal property of G/H, we get a unique G-equivariant morphism $\bar{\psi} : G/H \to X$ such that $\psi = \bar{\psi} \circ \pi$, i.e. $\bar{\psi}(gH) = \psi(g)$. We will prove that $\bar{\psi}$ is an isomorphism of ringed spaces – which will imply that G/H is a variety and that X is a categorical quotient, to complete the proof. We will use Zariski's main theorem and also 4.6 which tells us:

(*) For any variety Z, the morphism $\psi \times 1 : G \times Z \to X \times Z$ is open.

We obviously have that $\bar{\psi}$ is a continuous bijection. If $U \subseteq G/H$ is open, then $\bar{\psi}U = \psi(\pi^{-1}U)$ which is open by (*). Hence $\bar{\psi}$ is a homeomorphism of topological spaces. It remains to prove that $\bar{\psi}^* : \mathcal{O}_X(U) \to \mathcal{O}(\bar{\psi}^{-1}U)$ is an isomorphism for all open $U \subseteq X$. We need to show:

(**) for every regular function f on $V := \psi^{-1}U$ such that $f(gh) = f(g) \forall g \in V, h \in H$, there is a regular function F on U such that $F(\psi v) = f(v) \forall v \in V$.

Let $\Gamma = \{(g, f(g)) \mid g \in V\} \subseteq V \times \mathbb{A}^1$, $\Gamma' = (\psi \times 1)\Gamma \subseteq U \times \mathbb{A}^1$. Since Γ is closed in $V \times \mathbb{A}^1$,

$$(\psi \times 1)(V \times \mathbb{A}^1 - \Gamma) = U \times \mathbb{A}^1 - \Gamma'$$

is open in $U \times \mathbb{A}^1$ by (*), hence Γ' is closed in $U \times \mathbb{A}^1$. Let $\lambda : \Gamma' \to U$ be the morphism induced by the first projection. Clearly, λ is bijective. Note $\psi \circ pr_1 : \Gamma \to U$ factors as $\lambda \circ (\psi \times 1)$. The former is separable by the definition of a Chevalley quotient, so its differential is onto. Hence $d\lambda$ must be onto too. So λ is separable. Now we get that λ is birational using the separability just proved and 4.5(iii), just like we did in the proof of 4.6(iii).

Finally note that U is open in X, and X is smooth. So U is smooth. Now Zariski's main theorem gives that λ is an isomorphism Now look at $F := pr_2 \circ \lambda^{-1} : U \to \mathbb{A}^1$, which maps $u \in U$ to $f(v)$ where $\psi v = u$. This is a morphism. We are done. $\square$

We have now shown: for any closed subgroup H of G, the quotient $\pi : G \to G/H$ in the category of varieties exists. Indeed, it is given by any Chevalley quotient. In particular, G/H is a quasi-projective variety, and the morphism π is separable. Also note that

$$T_H(G/H) \cong \mathfrak{g}/\mathfrak{h}$$

canonically. Indeed, the map $\pi : G \to G/H$ is separable, so its differential is onto and has kernel equal to $\mathfrak{h}$ by dimension. So it induces the above isomorphism of tangent spaces.

Example 4.15. (1) Let $G = GL(V)$, $v_1, \ldots, v_n$ be a basis for V, and $H = stab_G(\langle v_1 \rangle)$. Then $X = G\langle v_1 \rangle = \mathbb{P}(V)$ is a Chevalley quotient of G by H. (We proved this above in greater generality!) So $G/H \cong \mathbb{P}^{n-1}$.

(2) Similarly, let $H = stab_G(\langle v_1, \ldots, v_d \rangle)$. Then $G\langle v_1, \ldots, v_d \rangle = G_d(V)$ is a Chevalley quotient, i.e. the Grassmann variety $G_d(V)$ is isomorphic to the quotient variety G/H.

(3) Finally let $H = stab_G(\langle v_1 \rangle \subset \cdots \subset \langle v_1, \ldots, v_k \rangle \subset \cdots \subset V)$, i.e. the stabilizer in G of the standard flag. So H is the group of all upper triangular matrices. In this case, $G/H \cong \mathscr{F}(V)$, the flag variety, because this is the G-orbit of the standard flag and the orbit map is separable.

(4) In all the above examples, the quotient variety G/H was a *projective variety.* Remember by construction it is always at least quasi-projective. Instead, let $H = stab_G(v_1)$. This time. $G/H \cong G.v_1$ which is $\mathbb{A}^n - \{0\}$, an open subset of an affine variety. This is neither affine nor projective.

(5) Let $G = GL_n$ and $H = O_n = \{g \in GL_n \mid g^T g = I\}$, assuming the characteristic is not 2. Let S be the set of all $n \times n$ symmetric matrices, an irreducible affine variety of dimension $\frac{1}{2}n(n+1)$. Let $S^\times$ be the invertible matrices in S, a principal open subset of S hence also affine of dimension $\frac{1}{2}n(n+1)$.

Let G act on S by $g.x = g^T xg$. Then $G_x = O_n$. So the orbit map induces, by the universal property of the quotient G/H, a bijective function $GL_n/O_n \to G.x$. By linear algebra, all non-degenerate symmetric bilinear forms are equivalent to the standard one, i.e. all invertible symmetric matrices can be written as $g^T g$ for some invertible matrix g. Hence, we have constructed a bijective morphism $GL_n/O_n \to S^\times$. In particular, $\dim GL_n - \dim O_n = \dim S^\times$, i.e. $\dim O_n = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$.

Moreover, to prove that $GL_n/O_n \cong S^\times$ we just need to show that the orbit map $g \mapsto g^T g$ is separable. Its differential is the map $X \mapsto X^T + X$. The tangent space to $S^\times$ at I is identified with S. Clearly any symmetric matrix can be written as $X^T + X$ (except in characteristic 2!). Therefore the orbit map is separable. Hence, the quotient variety $GL_n/O_n \cong S^\times$, which is an *affine variety.*

Exercise 4.16. (1) Let $H_1 < G_1, H_2 < G_2$ be closed subgroups of connected algebraic groups. Prove that $(G_1 \times G_2)/(H_1 \times H_2) \cong G_1/H_1 \times G_2/H_2$ as varieties.

(2) Prove that GL_{2n}/Sp_{2n} is isomorphic to the affine variety of all invertible $2n \times 2n$ skew symmetric matrices. Deduce that $\dim Sp_{2n} = n(2n+1)$. (In characteristic 2 a skew symmetric matrix means a symmetric matrix with zeros on the diagonal.)

4.4. **Normal subgroups.** Let G be an algebraic group. A *character* of G means a homomorphism $\chi : G \to \mathbb{G}_m$ of algebraic groups. We write $X(G)$ for the set of all characters of G. It has a natural structure of Abelian group: $(\chi + \chi')(g) = \chi(g)\chi'(g)$.

Example 4.17. (1) Let $G = SL_n$, or any other non-abelian simple algebraic group (remember, that means G has no closed connected normal subgroups other than 1 and itself). Then if $\chi : G \to \mathbb{G}_m$ is a character, $\ker \chi$ is a closed normal subgroup, hence is G since G is non-abelian. Hence, χ is the trivial character $g \mapsto 1$, and $X(G) = 0$.

(2) Let $G = \mathbb{G}_a$. Consider $\mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_m)$ just as varieties for now. It is $\cong \mathrm{Hom}(k[T, T^{-1}], k[T])$. You get such by mapping T to any invertible element. But the invertible elements in $k[T]$ are just the non-zero scalars. Therefore the only morphisms of varieties $\mathbb{G}_a \to \mathbb{G}_m$ are the constant ones, and of these only the one $x \mapsto 1$ is a group homomorphism. So $X(\mathbb{G}_a) = 0$.

(3) Let $G = \mathbb{G}_m$. Consider $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$ as varieties, i.e.

$$\mathrm{Hom}(k[T, T^{-1}], k[T, T^{-1}]).$$

They are all the maps sending T to an invertible element. The units in $k[T, T^{-1}]$ are the cT^n for $c \in k^\times$ and $n \in \mathbb{Z}$. Using in addition that it is a group homomorphism you get $c = 1$. This shows: $X(\mathbb{G}_m) \cong \mathbb{Z}$, the map sending $n \in \mathbb{Z}$ to the character $x \mapsto x^n$.

(4) $X(G \times H) \cong X(G) \oplus X(H)$. Hence, recalling $D_n \cong \mathbb{G}_m^n$, $X(D_n) \cong \mathbb{Z}^n$. Explicitly, the n-tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ corresponds to the character $diag(d_1, \dots, d_n) \mapsto d_1^{\lambda_1} \dots d_n^{\lambda_n}$ of D_n.

Exercise 4.18. (3) Prove that $X(G \times H) \cong X(G) \oplus X(H)$.

(4) Prove that $X(GL_n) \cong \mathbb{Z}$. Write down the isomorphism explicitly! (You should use the fact that SL_n is simple).

Let G be a linear algebraic group and suppose $\phi : G \to GL(V)$ is a representation of G on a finite dimensional vector space V, i.e. ϕ is a morphism of algebraic groups. For $\chi \in X(G)$, let

$$V_\chi := \{v \in V \mid \phi(g)v = \chi(g)v \forall g \in G\}.$$

This is called the χ-weight space of V. Let me remind of *linear independence of characters* (which is proved in the same way that you prove that different eigenspaces are linearly independent in linear algebra):

$$\sum_{\chi \in X(G)} V_\chi = \bigoplus_{\chi \in X(G)} V_\chi.$$

On the other hand, it will usually not be the case that V is the sum of all the weight spaces!!!

There is one very important case when V is the sum of its weight spaces, i.e.

$$V = \bigoplus_{\chi \in X(G)} V_\chi.$$

This is when the group G is the group D_n. For the elements of D_n are semisimple, hence their images in a representation are semisimple automorphisms of V (see 2.19). Moreover they commute. So you can simultaneously diagonalize V with respect to the endomorphisms coming from D_n. The group

$$D_n \cong (\mathbb{G}_m)^{\times n}$$

is the (algebraic) *n-dimensional torus*. (Why? Well perhaps this is convincing. A torus is a product of S^1's, where $S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$ – which is not an algebraic variety! The algebraic analogue of S^1 is simply all of $\mathbb{C}^\times$, i.e. $\mathbb{G}_m$. So an algebraic torus is a product of $\mathbb{G}_m$'s.)

Characters will play an important role later on. For now, let us prove:

Theorem 4.19. *Let G be an algebraic group, H a closed normal subgroup. Then, the variety G/H is affine, and the operations $(g_1H, g_2H) \mapsto g_1g_2H$ and $gH \mapsto g^{-1}H$ are morphisms of varieties.*

Proof. Let me first show that $(g_1H, g_2H) \mapsto g_1g_2H$ is a morphism. The map $G \times G \to G/H, (g_1, g_2) \mapsto g_1g_2H$ is a morphism that is constant on cosets of $H \times H$. Hence by the universal property of the quotients, we get induced a unique morphism $G \times G/H \times H \to G/H$. Now use Exercise (1) above: $G \times G/H \times H \cong G/H \times G/H$. The proof that $gH \mapsto g^{-1}H$ is a morphism easier...

So the main thing is to prove that the variety G/H is *affine*. By Chevalley's theorem, we can find a finite dimensional representation $\phi : G \to GL(V)$ and $v \in V$ such that

$$H = stab_G(\langle v \rangle), \mathfrak{h} = stab_{\mathfrak{g}}(\langle v \rangle).$$

Let $V' = \bigoplus_{\chi \in X(H)} V_\chi$ be the sum of all H-weight spaces in V. Note $v \in V'$. Moreover, H stabilizes each V_χ, and it is a normal subgroup of G, hence G must permute the V_χ amongst themselves. In particular, V' is invariant under G, so we may assume (replacing V by V') that $V = V'$.

Now let

$$W = \{f \in \mathrm{End}(V) \mid f(V_\chi) \subseteq V_\chi \text{ for all } \chi \in X(H)\}.$$

Note W is a vector space. Define a morphism of algebraic groups

$$\psi : G \to GL(W), \psi(g)f = \phi(g)f\phi(g)^{-1} \in \text{End}(V).$$

Let us compute $\ker\psi$: if $\psi(g) = \text{Id}$ then $\phi(g)$ commutes with all $f \in W$ hence (by Schur's lemma!) f's) $\phi(g)$ acts as scalars on each V_χ. Hence g stablizes $\langle v\rangle$, so $g \in H$.

So by the universal property of quotients, ψ induces a morphism

$$\lambda : G/H \to GL(W).$$

The image of λ is a closed – hence affine – subgroup of $GL(W)$. It just remains to prove that λ is an isomorphism onto its image. To do this, we just need to check that $d\psi_e$ is onto, or equivalently by dimensions that $\ker d\psi_e \subseteq \mathfrak{h}$. Suppose $X \in \ker d\psi_e$. Then, $d\psi_e(X)(f) = d\phi_e(X).f - f.d\phi_e(X) = 0$, so $d\phi_e(X)$ commutes with all $f \in W$. This shows that $d\phi_e(X)$ acts as a scalar on all V_χ's, in particular it stabilizes $\langle v\rangle$, hence $X \in \mathfrak{h}$. □

Now you can prove things like the "First isomorphism theorem" for algebraic groups:

Theorem 4.20. *Suppose that $\phi : G \to H$ is a separable morphism of algebraic groups (i.e. $d\phi_e$ is onto). Let $N = \ker\phi$. Then, ϕ induces an isomorphism $G/N \cong H$.*

Note in characteristic 0, separability is equivalent to ϕ being onto. But in characteristic p you definitely need the stronger assumption about the differentials. Here is an important example.

Let $G = H = GL_n$. Suppose $k = \overline{\mathbb{F}_p}$. Let $\phi : G \to H$ be the *Frobenius morphism* given by raising matrix entries to the pth power. This is a morphism of algebraic groups. Moreover, it is an isomorphism as abstract groups! But the differential $d\phi_e$ is the ZERO MAP. So ϕ is definitely not an isomorphism of algebraic groups.