3.5. **Separable morphisms.** Recall that a morphism \( \phi : X \to Y \) of irreducible varieties is called **dominant** if its image is dense in \( Y \). In the case of affine varieties, that is equivalent to \( \phi^* : k[Y] \to k[X] \) being injective. Hence, even for arbitrary irreducible \( X, Y \), the map \( \phi^* : k(Y) \to k(X) \) induced by a dominant morphism is injective. So we can view \( k(X) \) as an extension field of \( k(Y) \). The morphism \( \phi \) is called **separable** if \( k(X) \) is a separable extension of \( k(Y) \).

Go back to the case that \( X, Y \) are affine. The composite of \( \phi^* \) and \( d_X : k[X] \to \Omega_X \) is a derivation \( k[Y] \to \Omega_X \). So by the universal property of differentials, we get induced a \( k[Y]\)-module map
\[
\bar{\phi}^* : \Omega_Y \to \Omega_X
\]
such that \( d_X \circ \phi^* = \bar{\phi}^* \circ d_Y \).

Let \( x \in X, y = \phi(x) \). The \( k[X]\)-module \( k_x \) viewed as a \( k[Y]\)-module via \( \phi^* \) is \( k_y \). We used this before to see that \( \phi^* \) induced the linear map
\[
d\phi_x : T_x(X) = \text{Der}_k(k[X], k_x) \to T_y(Y) = \text{Der}_k(k[Y], k_y), D \mapsto D \circ \phi^*.
\]
Equivalently, we can view \( d\phi_x \) as the map
\[
d\phi_x : \text{Hom}_{k[X]}(\Omega_X, k_x) \to \text{Hom}_{k[Y]}(\Omega_Y, k_y), \theta \mapsto \theta \circ \bar{\phi}^*.
\]
Applying adjointness of tensor and hom, we can even view \( d\phi_x \) as a linear map
\[
\text{Hom}_k(\Omega_X(x), k) \to \text{Hom}_k(\Omega_Y(y), k).
\]

**Theorem 3.15.** Let \( \phi : X \to Y \) be a morphism of irreducible varieties.

(i) Assume that \( x \in X \) and \( y = \phi(x) \in Y \) are simple points and that \( d\phi_x \) is surjective. Then, \( \phi \) is a dominant separable morphism.

(ii) Assume \( \phi \) is a dominant separable morphism. Then the points \( x \in X \) with the property of (i) form a non-empty open subset of \( X \).

**Proof.** We may assume \( X \) and \( Y \) are affine and \( \Omega_X, \Omega_Y \) are free \( k[X] \)-resp. \( k[Y] \)-modules of rank \( d = \dim X \) resp. \( e = \dim Y \). In particular, \( X \) and \( Y \) are smooth.

The map \( \bar{\phi}^* : \Omega_Y \to \Omega_X \) of \( k[Y]\)-modules induces a homomorphism of free \( k[X]\)-modules
\[
\psi : k[X] \otimes_{k[Y]} \Omega_Y \to \Omega_X.
\]
We can represent \( \psi \) as a \( d \times e \) matrix \( A \) with entries in \( k[X] \), fixing bases for \( \Omega_X, \Omega_Y \). Suppose that \( d\phi_x \) is surjective. Then, \( A(x) \), which represents the dual map \( (d\phi_x)^* : \Omega_Y(y) \to \Omega_x(x) \), is injective hence a matrix of rank \( e \). Hence the rank of \( A \) is at least \( e \), hence equal to \( e \) since rank cannot be more than the number of columns. This shows that \( \psi \) is injective.

Hence \( \bar{\phi}^* \) is injective too. Since \( \Omega_X \) and \( \Omega_Y \) are free modules, this implies that \( \phi^* : k[Y] \to k[X] \) must be injective, so \( \phi \) is dominant. Moreover, injectivity of \( \psi \) implies injectivity of
\[
k(X) \otimes_{k[Y]} \Omega_Y \to k(X) \otimes_{k(X)} \Omega_X.
\]
This is the map $\alpha$ in the exact sequence
\[ k(X) \otimes_{k(Y)} \Omega_{k(Y)/k} \xrightarrow{\alpha} \Omega_{k(X)/k} \xrightarrow{\beta} \Omega_{k(Y)/k(Y)} \to 0. \]
Hence $k(X)$ is a separable extension of $k(Y)$ by the differential criterion for separability.

The proof of (ii) is similar to the proof of (ii) in 3.12. \qed

**Corollary 3.16.** Let $G$ be a connected algebraic group.

(i) If $X$ is a variety on which $G$ acts transitively, then $X$ is irreducible and smooth. In particular $G$ is smooth.

(ii) Let $\phi : X \to Y$ be a $G$-equivariant morphism between two varieties on which $G$ acts transitively. Then $\phi$ is separable if and only if $d\phi_x$ is surjective for some $x \in X$ which is if and only if $d\phi_x$ is surjective for all $x \in X$.

(iii) Let $\phi : G \to H$ be a surjective homomorphism of algebraic groups. Then, $\phi$ is separable if and only if $d\phi_e$ is surjective.

**Proof.** (i) Take $x \in X$. The orbit map $g \mapsto gx$ is onto, so $X$ is irreducible as $G$ is. Since at least one point of $x$ is simple, and the action is transitive, we see that all points of $x$ are simple, so $X$ is smooth.

(ii) This follows at once from the theorem.

(iii) Apply (ii) to $X = G, Y = G'$. \qed

3.6. **Lie algebras.** Notation: if $G$ is an algebraic group, let $\mathfrak{g}$ denote the tangent space $T_e(G)$ to $G$ at the identity. At the moment, $\mathfrak{g}$ is just a vector space of dimension $\dim G$ (we will see in a while that it has additional structure as a Lie algebra). For example, the tangent spaces to $GL_n$, $SL_n$, $Sp_{2n}$, $SO_n$, ... at the identity will be denoted $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_n$, ... At the moment these are all just vector spaces!

**Example 3.17.**

(1) First, $GL_n$. Then:

$$\mathfrak{gl}_n = \text{Der}_k(k[GL_n], k_e)$$

is the $n^2$ dimensional vector space on basis $\{e_{i,j} | 1 \leq i, j \leq n\}$ where $e_{i,j}$ is the point derivation

$$f \mapsto \frac{\partial f}{\partial T_{i,j}}(e).$$

Here, $T_{i,j}$ is the $ij$-coordinate function, and $k[GL_n]$ is the localization of the polynomial ring $k[T_{i,j}]$ at determinant. I will always identify the vector space $\mathfrak{gl}_n$ with the vector space of $n \times n$ matrices over $k$, so that $e_{i,j}$ is identified with the $ij$ matrix unit.

(2) $SL_n$. Since $SL_n = V(\det -1)$ inside of $GL_n$, $\mathfrak{sl}_n$ is a canonically embedded subspace of $\mathfrak{gl}_n$. Indeed, it will be all matrices

$$X = \sum_{i,j} a_{i,j} e_{i,j}$$
such that
\[ X(\det - 1) = 0. \]

But (calculate!)
\[ \sum_{i,j} a_{i,j}e_{i,j} \left( \sum_{w \in S_n} \text{sgn}(w)T_{1,w1} \cdots T_{n,wn} - 1 \right) = \sum_i a_{i,i}. \]

So \( \mathfrak{sl}_n \) is all matrices in \( \mathfrak{gl}_n \) of trace zero.

(3) \( \mathfrak{sp}_{2n} \). Recall that \( \mathfrak{sp}_{2n} = \{ x \in \mathfrak{gl}_n \mid x^t J x = J \} \) where
\[ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \]
as a closed subgroup of \( GL_{2n} \). Let \( T \) be the \( 2n \times 2n \) matrix with \( ij \) entry \( T_{i,j} \), the \( ij \)-coordinate function on \( GL_{2n} \). Then, \( \mathfrak{sp}_{2n} = V(T^tKT - J) \) (4n² polynomial equations written as one matrix equation!). Then \( \mathfrak{sp}_{2n} \) is the \( X \in \mathfrak{gl}_{2n} \) such that \( X(T^tJT - J) = 0 \). You calculate: this is exactly the condition that \( X^tJ + J X = 0 \). Now you can calculate \( \dim \mathfrak{sp}_{2n} \) by linear algebra...

(4) Similarly, \( \mathfrak{so}_n \) (in characteristic \( \neq 2 \)) is all \( X \in \mathfrak{gl}_n \) satisfying \( X^T + X = 0 \).

(5) Also recall that \( T_n \) is all upper triangular invertible matrices. The tangent space \( t_n \) will be all upper triangular matrices embedded into \( \mathfrak{gl}_n \). Similarly you can compute the tangent spaces of \( D_n, U_n \ldots \)

Okay, now let’s introduce on \( \mathfrak{g} \) the structure of a Lie algebra. It is convenient to go via an intermediate: \( L(G) \). This is defined to be the vector space
\[ L(G) = \{ D \in \text{Der}_k(k[G], k[G]) \mid D(\lambda_x f) = \lambda_x D(f) \text{ for all } f \in k[G], x \in G. \]
(Recall: \( (\lambda_x f)(g) = f(x^{-1}g) \)). We call \( L(G) \) the left invariant derivations because they are the derivations commuting with the left regular action of \( G \) on \( k[G] \).

Now, \( \text{Der}_k(k[G], k[G]) \) is a Lie algebra with operation being the commutator: \([D, D'] = D \circ D' - D' \circ D\). Obviously, if \( D \) and \( D' \) are left invariant derivations, so is \([D, D']\). Therefore \( L(G) \) is a Lie subalgebra of \( \text{Der}_k(k[G], k[G]) \).

**Lemma 3.18.** Let \( G \) be a connected algebraic group, \( \mathfrak{g} = T_e(G) \). The maps
\[ L(G) \rightarrow \mathfrak{g}, D \mapsto ev_x \circ D \]
and
\[ \mathfrak{g} \rightarrow L(G), X \mapsto \tilde{X}, \]
where \( \tilde{X} : k[G] \rightarrow k[G] \) is the derivation with \( (\tilde{X} f)(g) = X(\lambda_g^{-1} f) \) for all \( g \in G \), are mutually inverse isomorphisms.
Proof. We’d better first make sure the maps make sense, i.e. that $ev_e \circ D$ is a point derivation and that $\tilde{X}$ is a left invariant derivation. Only the last thing is tricky:

$$(\tilde{X}(\lambda_g f))(h) = X(\lambda_{h^{-1}} \lambda_g f) = X(\lambda_{h^{-1}} g h) = (\tilde{X} f)(g^{-1} h) = (\lambda_g (\tilde{X} f))(h).$$

Now let’s compute $ev_e \circ \tilde{X}$:

$$(ev_e \circ \tilde{X})(f) = (\tilde{X} f)(e) = X(f)$$

for all $f \in k[G]$, hence $ev_e \circ \tilde{X} = X$.

Finally, let’s compute $ev_e \circ D$ for a left invariant derivation $D$:

$$(ev_e \circ D(f))(g) = (ev_e \circ D)(\lambda_{g^{-1}} f) = (D(\lambda_{g^{-1}} f))(e) = (\lambda_{g^{-1}} (D f))(e) = (D f)(g).$$

Hence $ev_e \circ D = D$. □

Because of the lemma, we get on $\mathfrak{g}$ an induced structure as a Lie algebra, induced by the Lie algebra structure on $L(G)$:

$$[X, Y] := ev_e \circ [\tilde{X}, \tilde{Y}].$$

**Example 3.19.** Let’s compute the Lie algebra structure on $\mathfrak{gl}_n$. We need to work out $\tilde{e}_{i,j}$. Well, for $g \in G$,

$$(\tilde{e}_{i,j} T_{p,q})(g) = e_{i,j} (\lambda_{g^{-1}} T_{p,q}) = \frac{\partial}{\partial T_{i,j}} (\sum_r T_{p,r}(g) T_{r,q})(e) = \delta_{q,j} T_{p,i}(g).$$

Therefore $\tilde{e}_{i,j} T_{p,q} = \delta_{j,q} T_{p,i}$.

Now you can compute the commutator $[\tilde{e}_{i,j}, \tilde{e}_{k,l}]$: it acts on each $T_{p,q}$ in the same way as $\delta_{j,l} \tilde{e}_{i,l} - \delta_{i,l} \tilde{e}_{k,j}$. Hence we’ve worked out

$$[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - \delta_{i,l} e_{k,j}.$$ 

The right hand side is just the commutator $e_{i,j} e_{k,l} - e_{k,l} e_{i,j}$ of the matrix units. So in general:

$$[X, Y] = XY - YX$$

for $X, Y \in \mathfrak{gl}_n$ – the usual commutator as matrices!

**Lemma 3.20.** Let $H$ be a closed subgroup of $G$ and set $I = I(H)$. Then,

$$\mathfrak{h} = \{ X \in \mathfrak{g} \mid X I = 0 \} = \{ X \in \mathfrak{g} \mid \tilde{X} I \subseteq I \}.$$ 

In particular, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

**Proof.** That

$$\mathfrak{h} = \{ X \in \mathfrak{g} \mid X I = 0 \}$$

is just the definition of tangent space of a closed subvariety... If $XI = 0$, consider $\tilde{X} f$ for $f \in I$. Evaluating at $h \in H$,

$$(\tilde{X} f)(h) = X(\lambda_{h^{-1}} f)$$

which is zero as $\lambda_{h^{-1}} f$ is still in $I$ (as $H$ is a subgroup!). So $\tilde{X} f$ vanishes on $H$, i.e. is containing in $I$. This shows $\tilde{X} I \subseteq I$.

The opposite, namely that if $\tilde{X} I \subseteq I$ then $XI = \{0\}$ is easy.
Finally, observe that if \( D, D' \in L(G) \) satisfy \( DI \subseteq I, D'I \subseteq I \), then the commutator \([D, D']\) also does. This implies that \( \mathfrak{h} \) is a subalgebra of \( \mathfrak{g} \).  \( \square \)

Now we get the Lie algebra structure in all the above examples: \( \mathfrak{so}_n, \mathfrak{so}_n, \ldots \), since they are all just Lie subalgebras of \( \mathfrak{gl}_n \), where the Lie bracket is just the commutator as matrices.

The next lemma I'm going to leave to you to supply the proof (hint: its easier to rephrase things in terms of \( L(G) \) and \( L(H) \))...

**Lemma 3.21.** Let \( \phi : G \to H \) be a morphism of connected algebraic groups. Then \( d\phi : \mathfrak{g} \to \mathfrak{h} \) is a Lie algebra homomorphism.

3.7. **Some differential calculations.** I now want to compute differentials of various natural morphisms. In all cases, since an arbitrary algebraic group can be embedded as a closed subgroup of some \( GL_n \), it is enough to make the computation in the case of \( GL_n \), when we can be very explicit.

**Example 3.22.**

1. Let \( \mu : G \times G \to G \) be multiplication. Then, \( d\mu_{(e,e)} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \) is addition.

Proof: Suppose \( G = GL_n \). Recall \( \mathfrak{g} \oplus \mathfrak{g} \cong \text{Der}_k(k[G] \otimes k[G], k_{(e,e)}) \), the isomorphism mapping \( \langle X, Y \rangle \) to the map \( (f \otimes g) \mapsto X(f)g(e) + f(e)Y(g) \). By definition,

\[
(d\mu_{(e,e)}(e_{i,j}, e_{k,l}))(T_{r,s}) = (e_{i,j}, e_{k,l})(\sum T_{r,t} \otimes T_{t,s}) = \delta_{i,r}\delta_{j,s} + \delta_{k,s}\delta_{r,k} = (e_{i,j} + e_{k,l})(T_{r,s}).
\]

Hence \( d\mu_{(e,e)}(e_{i,j}, e_{k,l}) = e_{i,j} + e_{k,l} \), i.e. \( d\mu_{(e,e)} \) is addition.

2. Let \( i : G \to G \) be the inverse map. Then, \( di : \mathfrak{g} \to \mathfrak{g} \) is the map \( X \mapsto -X \).

Proof: Consider the composite \( G \to G \times G \to G, g \mapsto (g, i(g)) \mapsto gi(g) = e \). The composite is a constant function, so its differential is zero. But the differential of a composite is the composite of the differentials, so applying (1), \( 0 = d\text{id}_e + di \). The differential of the identity map is the identity map, so we are done.

3. Fix \( x \in G \). Let \( \text{Int} x : G \to G \) be the automorphism of algebraic groups \( g \mapsto xgx^{-1} \). The differential \( d(\text{Int} x)_e : \mathfrak{g} \to \mathfrak{g} \) is a Lie algebra automorphism. It is usually denoted \( \text{Ad} x : \mathfrak{g} \to \mathfrak{g} \).

For example, suppose \( G = GL_n \). Then, for a matrix \( X \in \mathfrak{gl}_n \), \( (\text{Ad} x)(X) = xXx^{-1} \), i.e \( \text{Ad} x \) is just the Lie algebra automorphism given by conjugation by \( x \). Hence, for \( H \) and closed subgroup of \( G \) and \( x \in H \), \( \text{Ad} x : \mathfrak{h} \to \mathfrak{h} \) is just conjugation by \( x \) too – it leaves the subspace \( \mathfrak{h} \) of \( \mathfrak{gl}_n \) invariant.

Proof: Let us compute \( (\text{Int} x)^*T_{i,j} \):

\[
((\text{Int} x)^*T_{i,j})(g) = T_{i,j}(xgx^{-1}) = (\mu^*(\mu^*T_{i,j}))(x, g, x^{-1})
\]

\[
= \sum_{k,l} T_{i,k}(x)T_{k,l}(g)T_{l,j}(x^{-1}).
\]
Hence:

\[(\text{Int } x)^*T_{i,j} = \sum_{k,l} x_{i,k} T_{k,l}(x^{-1})_{l,j}.\]

The \(ij\)-entry of \((\text{Ad } x)(X)\) is

\[(\text{Ad } x)(X)(T_{i,j}) = \sum_{k,l} x_{i,k} X(T_{k,l})(x^{-1})_{i,j}\]

which is the \(ij\)-entry of \(x X x^{-1}\).

(4) Here is a consequence of (3). For each \(x \in G\), \(\text{Ad } x : g \to g\) is an invertible linear map (even a Lie algebra automorphism) so you can think of Ad as a group homomorphism from \(G\) to \(GL(g)\) (or even to \(\text{Aut}(g)\) which is a closed subgroup of \(GL(g)\)). I claim that \(\text{Ad} : G \to GL(g)\) is a morphism of algebraic groups.

Proof: Embed \(G\) as a closed subgroup of some \(GL_n\). Then by (3), \(\text{Ad } x\) is conjugation by \(x\), which is clearly a morphism of varieties — since it is given by matrix multiplication and inversion which are polynomial operations.

(5) The image of \(\text{Ad} : G \to GL(g)\) is a closed connected subgroup of \(\text{Aut}(g)\), denoted \(\text{Ad } G\). It is interesting to consider the kernel of \(\text{Ad}\), a closed normal subgroup of \(G\). Obviously, every element of \(Z(G)\) belongs to \(\ker \text{Ad}\), i.e. \(Z(G) \subseteq \ker \text{Ad}\). I warn you that equality need not hold here. However it usually does, for example it always does in characteristic 0.

For example, if \(G = GL_n\), \(\ker \text{Ad} = Z(G)\) (check directly: \(\ker \text{Ad}\) is the invertible matrices which commute with all other matrices), the scalar matrices. In this case \(\text{Ad } G\) is the group known as \(PGL_n\). As an abstract group, \(PGL_n \cong GL_n/\{\text{scalars}\}\).

Similarly, if \(G = SL_n\), \(\ker \text{Ad}\) is the scalar matrices of determinant one, i.e. the matrices

\[
\begin{pmatrix}
\omega & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \omega
\end{pmatrix}
\]

where \(\omega\) runs over all \(n\)th roots of unity. The group \(\text{Ad } SL_n\) is known as \(PSL_n\). As an abstract group \(PSL_n \cong SL_n/\{\text{scalars}\}\).

Warning. In characteristic \(p|n\), \(Z(SL_n)\) is trivial. So \(PSL_n \cong SL_n\) as an abstract group, the isomorphism being the map \(g \mapsto \text{Ad } g\). However, \(SL_n\) is not isomorphic to \(PSL_n\) as an algebraic group: the problem is that \(\text{Ad}\) is a bijective morphism that is NOT an isomorphism of varieties, i.e. its inverse is not a morphism of varieties.

(6) Because of (4), we have a morphism \(\text{Ad} : G \to GL(g)\). So we can consider its differential again,

\[d \text{Ad} : g \to \mathfrak{gl}(g)\].
The map \( d\text{Ad} \) is usually denoted \( \text{ad} \). I claim that \( \text{ad} X \in \mathfrak{gl}(g) \) is the map \( Y \mapsto [X,Y] \) (in particular, \( \text{ad} X \) is a derivation of \( g \), so \( \text{ad} \) is a Lie algebra homomorphism \( g \to \text{Der}(g) \)). In other words, 

*The differential of \( \text{Ad} \) is \( \text{ad} \).*

The proof is so nasty I’m not going to type it in...

**Exercise 3.23.** (8) Here is an example to show that \( \ker \text{Ad} \) may be larger than \( Z(G) \) if \( \text{char} k = p > 0 \). Let \( G \) be the two dimensional closed subgroup of \( GL_2 \) consisting of all matrices of the form 

\[
\begin{pmatrix}
a & 0 \\
0 & a^p \ b
\end{pmatrix}
\]

where \( a \neq 0 \) and \( b \) are arbitrary elements of \( k \). Describe the Lie algebra \( g \) explicitly as a subspace of \( \mathfrak{gl}_2 \). Now compute \( Z(G) \) and \( \ker \text{Ad} \) and show that they are *not* equal.

(9) Consider the morphism \( \text{Ad} : SL_2 \to PSL_2 \). Both \( SL_2 \) and \( PSL_2 \) are three dimensional algebraic groups, so their Lie algebras are three dimensional too. Consider the differential \( \text{ad} : \mathfrak{sl}_2 \to \mathfrak{psl}_2 \). Using Example (8) above, compute the kernel of \( \text{ad} \), an ideal in the Lie algebra \( \mathfrak{sl}_2 \). Deduce by 3.16 that the morphism \( \text{Ad} \) is separable if and only if \( \text{char} k \neq 2 \). (So in characteristic 2 – when \( \text{Ad} \) is a bijective morphism – it is inseparable so it cannot be an isomorphism).

(10) Let \( \text{char} k = p > 0 \). Take \( X \in \mathfrak{g} = \text{Der}_k(k[G], k_e) \). Recall \( \mathfrak{g} \) is isomorphic to \( L(G) \), the left invariant derivations of \( k[G] \), via the map \( X \mapsto \tilde{X} \). Show that \((\tilde{X})^p\) is a left invariant derivation of \( k[G] \). Hence, there is a unique element \( X^{[p]} \in \mathfrak{g} \) with 

\[
\tilde{X}^{[p]} = (\tilde{X})^p.
\]

This gives an extra operation on \( \mathfrak{g} \) in characteristic \( p \), the map \( X \mapsto X^{[p]} \), which makes \( \mathfrak{g} \) into what is known as a *restricted Lie algebra*. Describe this map explicitly in the one dimensional cases \( G = \mathbb{G}_a \) and \( G = \mathbb{G}_m \).