

2. BASIC NOTIONS OF ALGEBRAIC GROUPS

Now we are ready to introduce algebraic groups and prove some of their basic properties.

2.1. Definition and first examples. An algebraic group is an affine variety G equipped together with morphisms of varieties $\mu : G \times G \rightarrow G$ and $i : G \rightarrow G$ that give the points of G the structure of a group (i.e. μ is multiplication and i is inverse satisfying the group axioms). In other words, an algebraic group is both a group and an affine variety, so that the group operations are morphisms with respect to the variety structure. (It is reasonable to consider algebraic groups that are not necessarily affine varieties so strictly speaking one should say “affine algebraic group” for the thing I have just defined. Since we will only ever meet affine algebraic groups I’ll drop the word affine...)

A closed subgroup H of an algebraic group G is a subgroup that is closed in the Zariski topology. Such subgroups are again algebraic groups in their own right.

A morphism $f : G \rightarrow H$ of algebraic groups is a morphism of varieties that is a group homomorphism too. The kernel of a morphism of algebraic groups is a closed, normal subgroup of G . But note we do not yet know that the image $f(G)$ of a morphism is a *closed* subgroup of H so we had better not talk about images of morphisms as algebraic groups yet.

- Example 2.1.**
- (1) The *additive group* \mathbb{G}_a is the group $(k, +)$, i.e. the affine variety \mathbb{A}^1 under addition.
 - (2) The *multiplicative group* \mathbb{G}_m is the group (k^\times, \cdot) , i.e. the principal open subset $\{x \in \mathbb{A}^1 \mid x \neq 0\}$ of \mathbb{A}^1 under multiplication.
 - (3) The group $GL_n = GL_n(k)$ is the group of all $n \times n$ invertible matrices over k . Obviously, the set of all $n \times n$ matrices can be identified with the affine space \mathbb{A}^{n^2} . Then GL_n is the principal open subset defined by the non-vanishing of determinant (which is a polynomial function in the matrix entries). Hence GL_n is an affine variety. Since the formulae for matrix multiplication and inversion are polynomials in the matrix entries and $1/\det$, the group structure maps are morphisms of varieties.
 - (4) The group $SL_n = SL_n(k)$ is the closed subgroup of GL_n defined by the zeros of the function $\det - 1$.
 - (5) The group D_n of invertible diagonal matrices is a closed subgroup of GL_n (defined by the zeros of which functions?). It is isomorphic to the direct product of n copies of \mathbb{G}_m .
 - (6) The group U_n of upper uni-triangular matrices in GL_n is another closed subgroup.
 - (7) The group T_n of all upper triangular invertible matrices in GL_n is yet another. It contains U_n as a closed normal subgroup, and D_n as a closed subgroup.

- (8) The orthogonal group $O_n = \{x \in GL_n \mid x^t x = 1\}$ is a closed subgroup of GL_n . But we had better exclude characteristic 2 when discussing this example...
- (9) The special orthogonal group $SO_n = O_n \cap SL_n$ is a normal subgroup of O_n of index 2 (remember: the characteristic is not 2!).
- (10) The symplectic group $Sp_{2n} = \{x \in GL_{2n} \mid x^t J x = J\}$ where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is another closed subgroup of GL_{2n} .

Well, that is a lot of examples. Let us go back and look at \mathbb{G}_a again. Its coordinate ring $k[\mathbb{G}_a]$ is the polynomial ring $k[T]$. The comorphisms μ^* and i^* give us algebra homomorphisms

$$\mu^* : k[T] \rightarrow k[T] \otimes k[T], \quad i^* : k[T] \rightarrow k[T].$$

You can work them out: $\mu^*(T) = T \otimes 1 + 1 \otimes T$ and $i^*(T) = -T$. Also let $\epsilon : k[T] \rightarrow k$ be the evaluation map at the identity element of \mathbb{G}_a , i.e. $\epsilon(T) = 0$. Then, μ^*, ϵ and i^* give the comultiplication, counit and antipode making the algebra $k[T]$ into a commutative *Hopf algebra*.

(Big aside: definition of Hopf algebra if you've never seen it before. A *coalgebra* is a vector space A together with linear maps $\Delta : A \rightarrow A \otimes A$ and $\epsilon : A \rightarrow F$, called the *comultiplication* and *counit* (or 'augmentation') respectively, such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\Delta \otimes 1} & A \otimes A \\ \uparrow 1 \otimes \Delta & & \uparrow \Delta \\ A \otimes A & \xleftarrow{\Delta} & A \\ k \otimes A & \xleftarrow{\sim} & A & \xrightarrow{\sim} & A \otimes k \\ \uparrow \epsilon \otimes 1 & & \parallel & & \uparrow 1 \otimes \epsilon \\ A \otimes A & \xleftarrow{\Delta} & A & \xrightarrow{\Delta} & A \otimes A \end{array} \quad (1)$$

$$\begin{array}{ccc} k \otimes A & \xleftarrow{\sim} & A & \xrightarrow{\sim} & A \otimes k \\ \uparrow \epsilon \otimes 1 & & \parallel & & \uparrow 1 \otimes \epsilon \\ A \otimes A & \xleftarrow{\Delta} & A & \xrightarrow{\Delta} & A \otimes A \end{array} \quad (2)$$

The first of these diagrams is the *coassociative law*.

If A and B are algebras, $A \otimes B$ is also naturally an algebra, with multiplication determined by the formula $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$ for $a_i \in A, b_i \in B$. A *bialgebra* A is a vector space A that is both an algebra, with multiplication μ and unit i , and a coalgebra, with comultiplication Δ and counit ϵ , such that $\Delta : A \rightarrow A \otimes A$ and $\epsilon : A \rightarrow k$ are algebra maps.

Finally, a *Hopf algebra* is a bialgebra A together with a linear map $S : A \rightarrow A$, called the *antipode*, such that the following diagrams commute:

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 i \otimes \epsilon \downarrow & & \downarrow 1 \otimes S \\
 A & \xleftarrow{\mu} & A \otimes A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 i \otimes \epsilon \downarrow & & \downarrow S \otimes 1 \\
 A & \xleftarrow{\mu} & A \otimes A
 \end{array}
 \tag{3}$$

End of big aside.)

In a similar way, $k[\mathbb{G}_m]$ is the localization of $k[T]$ at T , i.e. the algebra $k[T, T^{-1}]$ of Laurent polynomials. The Hopf algebra structure induced by the group structure of \mathbb{G}_m satisfies

$$\mu^*(T) = T \otimes T, i^*(T) = T^{-1}, \epsilon(T) = 1.$$

Similarly, *the coordinate ring of any algebraic group is an affine Hopf algebra*. Indeed, the categories of algebraic groups and affine Hopf algebras are contravariantly equivalent (just like the categories of affine varieties and affine algebras).

Exercise 2.2. (8) Consider the algebraic group GL_n . Let $T_{i,j}$ be the ij -coordinate function, i.e. the function on GL_n picking out the ij -entry of a matrix. Show that the coordinate ring $k[GL_n]$ is the localization of the polynomial algebra $k[T_{i,j} \mid 1 \leq i, j \leq n]$ at the function $\det = \det((T_{i,j})_{1 \leq i, j \leq n})$. Write down explicit formulae for the effect of the comorphisms μ^* and i^* on each $T_{i,j}$. Finally, what is the counit that makes $k[GL_n]$ into a Hopf algebra?

2.2. Linear algebraic groups. A *linear algebraic group* is a closed subgroup of some GL_n . We wish to prove the following theorem, which shows that GL_n plays the role for algebraic groups that the symmetric group S_n plays in Cayley's theorem about finite groups.

Theorem 2.3. *Every (affine) algebraic group is linear, i.e. is isomorphic to a closed subgroup of GL_n .*

To prove the theorem, we need to find a finite dimensional vector space, and the place to look is inside the coordinate ring $k[G]$ of G .

Given $g \in G$, there is a linear map

$$\rho_g : k[G] \rightarrow k[G], f \mapsto \rho_g f$$

where $(\rho_g f)(h) = f(hg)$. This gives us a group homomorphism

$$\rho : G \rightarrow GL(k[G])$$

(a homomorphism just as abstract groups: $GL(k[G])$ is not an algebraic group since $k[G]$ is usually infinite dimensional). We call ρ *right translation of functions*. Similarly, there is the *left translation of functions* $\lambda : G \rightarrow GL(k[G])$ where where $(\lambda_g f)(h) = f(g^{-1}h)$.

Lemma 2.4. *Given any finite dimensional subspace V of $k[G]$, there is a finite dimensional subspace W of $k[G]$ containing V that is invariant under all right translations ρ_x ($x \in G$).*

Proof. It suffices to prove this in the case V is one dimensional, spanned by $f \in k[G]$ say. Let W be the subspace of $k[G]$ spanned by all $\rho_g f$ for all $x \in G$. We need to show that W is finite dimensional.

Write $\mu^* f = \sum_{i=1}^n f_i \otimes g_i \in k[G] \otimes k[G]$. Let X be the finite dimensional subspace of $k[G]$ spanned by all f_i . Now consider $x \in G$. We have that $(\rho_x f)(h) = f(hx) = (\mu^* f)(h, x) = \sum_{i=1}^n f_i(h)g_i(x)$. Hence, $\rho_x f = \sum_{i=1}^n g_i(x)f_i$. Hence $\rho_x f \in X$. Hence $W \leq X$ and W is finite dimensional. \square

Now to prove the theorem, let G be an algebraic group. Choose linearly independent generators f_1, \dots, f_n for the coordinate ring $k[G]$. Applying the lemma, we may assume (adding finitely many more generators if necessary) that the span E of the f_i is invariant under all right translations. Now consider the restriction

$$\psi : G \rightarrow GL(E), x \mapsto \rho_x|_E$$

of ρ . This is a group homomorphism, and $GL(E)$ is now an algebraic group since E is finite dimensional.

To see that ψ is a morphism of varieties, write $\mu^* f_i = \sum_j m_j \otimes n_{i,j}$ with the m_j 's linearly independent. As in the proof of the lemma, $\rho_x f_i = \sum_j n_{i,j}(x)m_j$. Since this lies in E for all $x \in G$, we see that each m_j lies in E . So we may assume in fact that $m_j = f_j$ for each j , i.e.

$$\mu^* f_i = \sum_j f_j \otimes n_{i,j}.$$

Hence, $\rho_x f_i = \sum_j n_{i,j}(x)f_j$, which shows that the coordinates of the matrix $\rho_x|_E \in GL(E)$ with respect to the basis f_1, \dots, f_n are the $n_{i,j}$. Hence ψ is a morphism of varieties.

Next we show that ψ is injective. Indeed,

$$f_i(x) = f_i(ex) = \sum_j f_j(e)n_{i,j}(x),$$

so $f_i = \sum_j f_j(e)n_{i,j}$. If $\psi(x) = e$, then $n_{i,j}(x) = \delta_{i,j}$ so $f_i(x) = f_i(e)$ for all i so $x = e$.

The same equation shows that the $n_{i,j}$ generate $k[G]$, hence since $n_{i,j} = \psi^*(T_{i,j})$, the comorphism $\psi^* : k[GL(E)] \rightarrow k[G]$ is surjective. Hence the image $\psi(G)$ is a closed subgroup of $GL(E)$. We are done.

2.3. First properties. The results in this subsection are absolutely fundamental for everything else that follows...

Lemma 2.5. *Let G be an algebraic group.*

- (i) The identity element $e \in G$ belongs to a unique irreducible component G^0 of G .
- (ii) G^0 is a closed normal subgroup of G of finite index, and the irreducible components of G are the cosets of G^0 .
- (iii) Any closed subgroup of G of finite index contains G^0 .

Proof. (i) Let X and Y be two irreducible components containing e . Then the closure of $XY = \mu(X \times Y)$ is irreducible too since μ is a morphism. But \overline{XY} contains both X and Y , so $X = \overline{XY} = Y$.

(ii) The argument in (i) also shows that G^0 is closed under multiplication. It is also closed under the inverse map i since $i(G^0)$ is an irreducible component containing e . Hence, G^0 is a subgroup of G . It is even normal, since for $g \in G$, gG^0g^{-1} is an irreducible component containing e . Finally, all cosets xG^0 of G^0 are also irreducible components of G , in particular G^0 has finite index in G .

(iii) Finally let H be a closed subgroup of G of finite index. Then H^0 is a closed subgroup of finite index of G^0 . So H^0 is both open and closed in G^0 . But G^0 is irreducible, hence it is connected, so this means that $H^0 = G^0$. \square

The lemma shows that for algebraic groups, the irreducible components are disjoint, hence coincide with the connected components of the topological space G . Because of this, people usually talk just about connected components of an algebraic group (not irreducible components). The normal subgroup G^0 is referred to as the *identity component* of G .

Also note at this point that all components of G are isomorphic as varieties to G^0 , in particular they all have the same dimension. Because of this, it is reasonable to talk about $\dim G$ (meaning $\dim G^0$) even if G is not connected. For example, G is finite if and only if $\dim G = 0$, $\dim GL_n = n^2, \dots$

Lemma 2.6. *Let U and V be dense open subsets of G . Then $G = UV$.*

Proof. Let $x \in G$. Then xV^{-1} and U are dense open subsets. Since they are dense, they must both intersect G^0 , so that $xV^{-1} \cap G^0$ and $U \cap G^0$ are dense open subsets of G^0 . But G^0 is irreducible, hence xV^{-1} and U in fact intersect each other at some point $u \in U$. But then $u = xv^{-1}$ for some $v \in V$, hence $x = uv$. \square

Lemma 2.7. *Let H be a (not necessarily closed) subgroup of G .*

- (i) Its closure \overline{H} is a subgroup of G .
- (ii) If H contains a non-empty open subset of \overline{H} , then H is closed.

Proof. (i) Let $x \in H$. Since multiplication by x is a homeomorphism, $x\overline{H} = \overline{xH} = \overline{H}$. This shows that $H\overline{H} \subseteq \overline{H}$. Now let $x \in \overline{H}$. Then $Hx \subseteq \overline{H}$ by the previous paragraph, hence $\overline{H}x = \overline{Hx} \subseteq \overline{H}$. Hence \overline{H} is closed under multiplication. Also $(\overline{H})^{-1} = \overline{H^{-1}}$ so we get that \overline{H} is a subgroup.

(ii) Now suppose that H contains a non-empty open subset U of \overline{H} . Then H is also open in \overline{H} , since H is a union of translates of U . So we get from 2.6 that $H = HH = \overline{H}$. \square

Corollary 2.8. *Let $\phi : G \rightarrow H$ be a morphism of algebraic groups. Then its image $\phi(G)$ is a closed subgroup of H .*

Proof. By 1.28, $\phi(G)$ contains a non-empty open subset of its closure. Hence it is closed by 2.7(ii). \square

Finally, we end with a very useful, if nasty-looking, result:

Theorem 2.9. *Let $(X_i, \phi_i)_{i \in I}$ be a family of irreducible varieties and morphisms $\phi_i : X_i \rightarrow G$ such that $e \in \phi_i(X_i)$ for all i . Let H be the smallest subgroup of G containing each $\phi_i(X_i)$. Then:*

- (i) H is closed and connected;
- (ii) $H = Y_{a_1}^{\epsilon_1} \dots Y_{a_n}^{\epsilon_n}$ (set-wise product) for some $a_1, \dots, a_n \in I$ and $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$.

Proof. WLOG each of the sets Y_i^{-1} occur among the Y_j . Note for each $a = (a_1, \dots, a_n) \in I^n$, $Y_a = Y_{a_1} \dots Y_{a_n}$ is irreducible. Hence $\overline{Y_a}$ is irreducible too. Obviously, $Y_a Y_b = Y_{(a,b)}$, and now arguing as in the proof of 2.7(i) you show that $\overline{Y_a} \overline{Y_b} \subseteq \overline{Y_{(a,b)}}$.

Now choose the tuple a such that $\dim \overline{Y_a}$ is maximal. As $e \in Y_a$, we have for any b that $\overline{Y_a} \subseteq \overline{Y_a} \overline{Y_b} \subseteq \overline{Y_{(a,b)}}$. Equality holds by dimension, hence $\overline{Y_b} \subseteq \overline{Y_a}$ for every b and also $\overline{Y_a}$ is closed under multiplication.

Now we have shown that $\overline{Y_a}$ is a group. Since Y_a contains an open subset of $\overline{Y_a}$ by 1.28, we get that $\overline{Y_a} = Y_a Y_a$ from 2.6. Therefore $H = \overline{Y_a}$ and we have proved (i) and (ii). \square

Here are some applications:

- (i) Assume that $(G_i)_{i \in I}$ is a family of closed connected subgroups of G . Then, the subgroup H generated by them all is closed and connected, and moreover $H = G_{a_1} \dots G_{a_n}$ for some $a_1, \dots, a_n \in I$.
- (ii) The groups Sp_{2n} and SO_n (in characteristic $\neq 2$) are connected. Incidentally, SO_n has index two in O_n , hence it is the identity component of O_n .

Proof: This needs to know a little group theory about symplectic and orthogonal groups (which is covered in Kantor's course on Classical groups, or in this course but much later on...) For example, in the case of Sp_{2n} , this group is generated by so-called *transvection subgroups*. Let V be the natural $2n$ -dimensional vector space that Sp_{2n} acts on, with symplectic bilinear form (\cdot, \cdot) preserved by the group. A transvection subgroup is a subgroup consisting of all transformations of the form

$$\alpha_{a,t} : v \mapsto v + t(v, a)a$$

for $t \in k, 0 \neq a \in V$. The resulting subgroup $G_a = \{\alpha_{a,t} \mid t \in k\}$ is a closed subgroup of G isomorphic to \mathbb{G}_a , hence is connected. Since these generate all of Sp_{2n} (which is a theorem in group theory only) you get that Sp_{2n} is connected by (i).

- (iii) Let H and K be closed subgroups of G with H connected. Then, the commutator group (H, K) (generated by all commutators $[h, k] = hkh^{-1}k^{-1}$ with $h \in H, k \in K$) is closed and connected.

Proof: Take the index set I in the theorem to be K and the maps $\phi_k : H \rightarrow G$ to be the maps $h \mapsto xkx^{-1}k^{-1}$ for $k \in K$.

- (iv) Recall the definition of the derived series

$$G = G^{(0)} \geq G^{(1)} \geq \dots$$

of a group G : $G^{(0)} = G, G^{(i+1)} = (G^{(i)}, G^{(i)})$. The group G is then called solvable if $G^{(i)} = \{e\}$ for some i . In case G is a connected algebraic group, each of the derived subgroups are closed, connected normal subgroups of G . Therefore either $G^{(i)} = G^{(i+1)}$ – and the derived series has stabilized – or $\dim G^{(i+1)} < \dim G^{(i)}$. Thus we see that for algebraic groups the derived series stabilizes after finitely many terms; we will use this later on in particular in developing a nice theory of solvable algebraic groups. Similar remarks apply to nilpotent algebraic groups replacing the derived series with the descending central series.

Exercise 2.10. (9) Show that $\phi(G^{(0)}) = \phi(G)^0$, for ϕ a morphism of algebraic groups.

- (10) Let G be the group U_n of all upper uni-triangular matrices. Let $G^0 = G$ and $G^{i+1} = (G, G^i)$ (thus defining the descending central series of G). Describe the group G^i for each $i \geq 0$. Hence check directly (i.e. without applying 2.9) that it is a closed, connected subgroup of G . What is its dimension? Now prove that the group T_n of all upper triangular invertible matrices is a solvable algebraic group.