1.5. **Products.** Now let us work in the category of affine varieties. There is a general notion of product in any category: for example a product $X \times Y$ of affine varieties should be an affine variety together with morphisms $p_1 : X \times Y \to X, p_2 : X \times Y \to Y$ such that: for any other affine variety $Z$ and maps $q_1 : Z \to X, q_2 : Z \to Y$, there is a unique morphism $r : Z \to X \times Y$ such that $q_i = p_i \circ r$.

**Lemma 1.14.** Products exist in the category of affine varieties.

**Proof.** We have seen that there is a contravariant equivalence between the category of affine varieties and the category of affine algebras. Therefore it suffices to show that coproducts exist in the category of affine algebras. We already know coproducts exist in the category of all algebras: the coproduct of $A$ and $B$ is $A \otimes_k B$, with the maps $A \to A \otimes_k B, a \mapsto a \otimes 1$ and $B \to A \otimes_k B, b \mapsto 1 \otimes b$. Therefore we will be done if we can check:

---

Let $A$ and $B$ be affine algebras. Then $A \otimes_k B$ is an affine algebra.

To prove this, let $\sum_{i=1}^n a_i \otimes b_i$ be a nilpotent element of $A \otimes_k B$. We may assume the $b_i$ are linearly independent. Let $f : A \to k$ be a morphism. Then, $\sum_{i=1}^n f(a_i)b_i$ is nilpotent, hence $f(a_i) = 0$ for all $i$ by the independence of $b_i$'s. This shows that $a_i$ lies in every maximal ideal of $A$. So $V(a_i)$ contains every point of the corresponding algebraic set. So $V(a_i)$ contains all of the corresponding algebraic set. So $I(V(a_i)) = \sqrt{(a_i)} = (0)$ hence $a_i$ is 0 as $A$ is reduced.

Therefore it makes sense to write $X \times Y$ for affine varieties $X$ and $Y$. It is again an affine variety, with $k[X \times Y] = k[X] \otimes_k k[Y]$. Note as a set, $X \times Y$ is just the Cartesian product, and the projection maps $p_1, p_2$ in the definition of product are just the obvious projections. How would you prove this?

**Exercise 1.15.**

1. Arguing like in the above lemma, show that if $A$ and $B$ are integral domains, so is $A \otimes_k B$. Deduce that the product of two irreducible affine varieties is again irreducible.

2. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain. Therefore the fact that $k$ is algebraically closed is fundamental in the previous exercise.

1.6. **Varieties.** We are nearly ready to give the general definition of an algebraic variety. To start with, define a **prevariety** to be a quasi-compact ringed space $(X, \mathcal{O}_X)$ such that every point of $X$ has an open neighbourhood $U$ with the property that $(U, \mathcal{O}_X|_U)$ is an affine variety. (Such a $U$ is called an affine open subset of $X$). Note prevarieties are Noetherian topological spaces. A morphism of prevarieties means the same as a morphism of ringed spaces.

**Lemma 1.16.** Products exist in the category of prevarieties.

**Proof.** Here is the construction of the product $X \times Y$ of two prevarieties $X$ and $Y$. As a set, $X \times Y$ is the Cartesian product of $X$ and $Y$. We need to define a topology and a sheaf of functions...
Cover $X = \bigcup_{i=1}^{m} U_i, Y = \bigcup_{j=1}^{n} V_j$ for affine opens $U_i, V_j$. Then $X \times Y$ is covered by the $U_i \times V_j$. These are affine varieties. Now declare that $U \subseteq X \times Y$ is open if $U \cap (U_i \times V_j)$ is open in $U_i \times V_j$ for all $i, j$. Call a function $f$ defined in an open neighbourhood $U$ of $x \in U_i \times V_j$ regular if its restriction to $U \cap (U_i \times V_j)$ is regular at $x$ in the old sense for affine varieties. This gives the sheaf of functions.

Now finally call a prevariety $X$ a variety if the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$. For example, any affine variety is a variety: if $X$ is affine with coordinate ring $k[X]$, the diagonal $\Delta_X$ is the set of common zeros of the ideal $I = (f \otimes 1 - 1 \otimes f \mid f \in k[X]) \subset k[X] \otimes k[X]$, hence it is closed.

This last condition – that $\Delta_X$ is closed – is called the separation axiom. It is some sort of substitute for the Hausdorff property when working with the Zariski topology: indeed a topological space $X$ is Hausdorff if and only if $\Delta_X$ is closed in $X \times X$ for the product topology.

Here are some of the consequences of the separation axiom:

**Lemma 1.17.** Let $X, Y$ be a varieties.

(i) If $\phi : X \to Y$ is a morphism then its graph $\{(x, \phi(x)) \mid x \in X\}$ is closed in $X \times Y$.

(ii) If $\phi, \psi : X \to Y$ are two morphisms which coincide on a dense subset of $X$, then $\phi = \psi$.

*Proof.* For (ii), note that $\{x \in X \mid \phi(x) = \psi(x)\}$ is the inverse image of $\Delta_Y$ under the continuous map $x \mapsto (\phi(x), \psi(x))$. Therefore it is closed, and dense, therefore is all of $X$. Part (i) is similar, considering the continuous map $X \times Y \to Y \times Y, (x, y) \mapsto (x, \phi(y))$. \qed

**Example 1.18.**

(i) Suppose $f(x)$ and $g(x)$ are polynomials which are equal for infinitely many values of $x \in k$. View them as morphisms $\mathbb{A}^1 \to \mathbb{A}^1$; then they agree on a dense subset of $k$ so they are equal everywhere.

(ii) Here is an example of a prevariety that is not a variety: the affine line with a point doubled. It cannot possibly be a variety because the two maps $\mathbb{A}^1 \to X$ sending $\mathbb{A}^1$ to one of the affine lines or the other are two morphisms that agree on $\mathbb{A}^1 - \{0\}$, a dense subset, but that are not equal.

(iii) In a similar way we can make $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ into a prevariety. In this case it is even a variety; the easiest way to prove this is to appeal to the lemma 1.21 below.

We have one more job to do in this generality: we need to understand how to make certain subsets of algebraic varieties into algebraic varieties in their own right.

We can do this right away for open subsets: if $X$ is a variety and $U$ is open, we have constructed the ringed space $(U, \mathcal{O}_U)$ where $\mathcal{O}_U = \mathcal{O}_X|_U$. Now
cover \( X = \bigcup_{i=1}^{n} U_i \) by affine opens \( U_i \). Each \( U \cap U_i \) is open in \( U_i \), therefore can be written as a union \( U = \bigcup_{j=1}^{m} V_{ij} \) for principal open subsets \( V_{ij} \) of \( U_i \). Principal open subsets of affine varieties are affine, so the \( V_{ij} \) are affine. Hence, \( U \) has a finite affine open cover, so it is a prevariety. Finally, \( \Delta_U \) is closed as its \( \Delta_X \cap (U \times U) \) and the topology on \( U \times U \) is the subspace topology.

**Exercise 1.19.** (3) Consider the open subset \( X = \mathbb{A}^2 - \{(0,0)\} \). Give \( X \) the induced variety structure. Prove that \( X \) is not affine.

What we would really like is to make closed subsets of a variety into varieties too.

To start with, let \((X, \mathcal{O}_X)\) be a ringed space and \( Y \subset X \) an arbitrary closed subset. Give \( Y \) the subspace topology, and for \( U \) open in \( Y \), define \( \mathcal{O}_X|_Y(U) \) to be all functions on \( U \) with the following property: there exists an open covering \( U \subseteq \bigcup U_\alpha \) of \( U \) by opens in \( X \) and elements \( f_\alpha \in \mathcal{O}_X(U_\alpha) \) for each \( \alpha \) such that \( f_\alpha|_{U \cap U_\alpha} = f|_{U \cap U_\alpha} \). This makes \((Y, \mathcal{O}_X|_Y)\) into a ringed space in its own right (and generalizes the notion of the restriction of a sheaf of functions to an open subset we had before).

This is at least the sensible notion for closed subsets of affine varieties (which we already understand as varieties):

**Lemma 1.20.** Let \( X \) be an affine variety with coordinate ring \( k[X] \) and \( Y = V(I) \) be a closed subset. Then \((Y, \mathcal{O}_X|_Y)\) is again an affine variety with coordinate ring \( k[Y] = k[X]/\langle I \rangle \) (which is of course what we expected).

**Proof.** The main thing is to show that the restriction map

\[
k[X] = \mathcal{O}_X(X) \to \mathcal{O}_X|_Y(Y)
\]

is surjective – this shows that \( \mathcal{O}_X|_Y(Y) = k[Y] \) and given this everything else is routine.

Let \( f \in \mathcal{O}_X|_Y(Y) \). Arguing as in the proof of 1.10, we can find \( a_i, f_i \in k[X] \) such that \( Y \subseteq D(f_1) \cup \cdots \cup D(f_n) \), \( f = \frac{a}{f_1} \) on \( D(f_i) \cap Y \), and \( a_1 f_j = a_j f_i \) on \( Y \). Now, \( X - Y \) is open so we can represent it as \( D(h_1) \cup \cdots \cup D(h_m) \).

By the Nullstellensatz, \( k[X] \) is generated by the \( f_i \) and \( h_j \)'s, so we can write

\[
1 = c_1 f_1 + \cdots + c_n f_n + d_1 h_1 + \cdots + d_m h_m.
\]

Each \( h_j \) vanishes on \( Y \), so

\[
f|_{D(f_j) \cap Y} = \sum_i c_i f_i \frac{a_j}{f_j} = \sum_i c_i a_i.
\]

This shows that \( f \) is restriction to \( Y \) of \( \sum_i c_i a_i \in k[X] \), giving the required surjectivity. \( \square \)

Now let \((X, \mathcal{O}_X)\) be an arbitrary variety and let \( Y \) be a closed subset. We want to show that \((Y, \mathcal{O}_Y)\) is a variety, where \( \mathcal{O}_Y = \mathcal{O}_X|_Y \). Cover \( X = \bigcup_{i=1}^{n} U_i \) by affine opens. Then \( Y = \bigcup_{i=1}^{n} (Y \cap U_i) \), and \( Y \cap U_i \) is affine
since its a closed subset of \( U_i \) (use 1.20!). Hence, \((Y, \mathcal{O}_Y)\) is a prevariety, and its obvious that \( \Delta_Y \) is closed...

To finish with, here is the useful lemma promised in Example above.

**Lemma 1.21.** Suppose that \( X \) is a prevariety with an affine open cover \( X = \bigcup_{i=1}^n U_i \). Then, \( X \) is a variety if and only if for every pair \( i,j \) the intersection \( U_i \cap U_j \) is affine open and the images under restriction of \( \mathcal{O}_X(U_i) \) and \( \mathcal{O}_X(U_j) \) in \( \mathcal{O}_X(U_i \cap U_j) \) generate \( \mathcal{O}_X(U_i \cap U_j) \).

**Proof.** Suppose that \( X \) is a variety and take affine opens \( U, V \). We have that \( \Delta_X \cap (U \times V) \) is closed in \( U \times V \). The map \( i : X \to \Delta_X \) induces an isomorphism \( U \cap V \cong \Delta_X \cap (U \times V) \) of ringed spaces. Hence \( U \cap V \) is affine, as the latter is. Moreover, the regular functions on \( \Delta_X \cap (U \times V) \) are restrictions of regular functions on \( U \times V \), i.e. of elements of \( k[U] \otimes k[V] \), so we are done since the latter algebra is generated by \( k[U] \otimes 1 \) and \( 1 \otimes k[V] \).

Conversely, consider for each pair \( i, j \) the intersection \( Y = \Delta_X \cap (U_i \times U_j) \). The map \( i : X \to \Delta_X \) again induces an isomorphism

\[
U_i \cap U_j \cong Y
\]

of ringed spaces. Hence by assumption \( Y \) is an affine variety and its coordinate ring is the image under restriction of \( \mathcal{O}_{X \times X}(U_i \times U_j) = k[U_i] \otimes k[U_j] \). Hence \( Y \) is closed in \( U_i \times U_j \) (see 1.27(i) below for the proof). Since this is true for all \( i, j \) this implies that \( \Delta_X \) is closed in \( X \times X \), as required. \( \square \)

1.7. **Projective varieties.** Now we can introduce the most important of the non-affine varieties. In fact in this course, all varieties we shall ever encounter can be constructed as open subvarieties of projective varieties.

Let us start by defining \( \mathbb{P}^n \). As a set, this is the set of all lines in \( k^{n+1} \), i.e. its \( k^{n+1} - \{0\} / \sim \) where \( x \sim y \) if \( x = cy \) for a non-zero scalar \( c \). We will denote the **point** of \( \mathbb{P}^n \) represented by the vector \( 0 \neq x \in k^{n+1} \) by

\[
[x_0, x_1, \ldots, x_n].
\]

So:

\[
[x_0, \ldots, x_n] = [y_0, \ldots, y_n]
\]

if and only if \( x_i = cy_i \) for all \( i \), where \( c \) is a non-zero scalar. The \( x_i \) are called **homogeneous coordinates** for \( x \).

For \( 0 \leq i \leq n \), set

\[
U_i = \{ [x_0, \ldots, x_n] \in \mathbb{P}^n \ | \ x_i \neq 0 \}.
\]

Define a bijection

\[
\phi_i : U_i \to \mathbb{A}^n, \quad \phi_i([x_0, \ldots, x_n]) = (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i).
\]

Note for \( i \neq j \), \( \phi_i(U_i \cap U_j) \) is a principal open subset of \( \mathbb{A}^n \), defined by the non-vanishing of the \( j \)th coordinate function.

Now make \( \mathbb{P}^n \) into a topological space by declaring a subset \( U \) is open if and only if \( U \cap U_i \) is open for all \( i \). Check: the subspace topology on each \( U_i \) is exactly the Zariski topology on \( \mathbb{A}^n \) lifted through \( \phi_i \).
Next make $\mathbb{P}^n$ into a ringed space by defining a function $f$ on an open set $U$ to be regular if $f|_{U \cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$ for each $i$, where $\mathcal{O}_{U_i}$ is the sheaf of functions on $U_i$ obtained by lifting the sheaf of regular functions on $\mathbb{A}^n$ through $\phi_i$. Check: this defines a sheaf of functions $\mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$, and moreover $\mathcal{O}_{\mathbb{P}^n}|_{U_i} = \mathcal{O}_{U_i}$ for each $i$.

Hence, $\mathbb{P}^n$ is a prevariety as the $U_i$'s (with the induced structures as ringed spaces) are affine, isomorphic to $\mathbb{A}^n$. To check finally that it is a variety, it just remains to apply 1.21: we need to show that $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ is generated by the restrictions of functions in $\mathcal{O}_{\mathbb{P}^n}(U_i)$ and $\mathcal{O}_{\mathbb{P}^n}(U_j)$. Writing $T_i/T_j$ for the polynomial function $x \mapsto x_i/x_j$ on $U_j$,

$$k[U_i] = k[T_0/T_i, \ldots, T_j/T_i, \ldots, T_n/T_i],$$

$$k[U_j] = k[T_0/T_j, \ldots, T_i/T_j, \ldots, T_n/T_j].$$

Clearly the restrictions of these generate the localization of $k[U_i]$ at $T_j/T_i$ (i.e., the localization of $k[U_j]$ at $T_i/T_j$), which is $k[U_i \cap U_j]$. Done.

**Definition.** A **projective variety** is a variety that is isomorphic to a closed subvariety of some $\mathbb{P}^n$. A **quasi-projective variety** is a variety that is isomorphic to an open subvariety of some projective variety.

We end with a description of the closed sets in $\mathbb{P}^n$. Let $S = k[T_0, \ldots, T_n]$ be the polynomial algebra. An ideal $I$ of $S$ is called homogeneous if it is generated by homogeneous polynomials. For such ideals, we can consider $V(I) = \{x = [x_0, \ldots, x_n] \in \mathbb{P}^n \mid f(x) = 0 \text{ for all homogeneous } f \in I\}$ (which makes sense independent of the choice of representative for $x$).

**Lemma 1.22.** The closed sets of $\mathbb{P}^n$ are exactly the $V(I)$ for homogeneous ideals $I$ in $S$.

**Proof.** If $I$ is a homogeneous ideal, all $V(I) \cap U_i$ are closed in $U_i$, hence $V(I)$ is closed in $\mathbb{P}^n$.

Conversely, let $U$ be an open subset of $\mathbb{P}^n$, i.e. each $U \cap U_i$ is open in $U_i$. If $(I_\alpha)_{\alpha \in A}$ is a family of homogeneous ideals and $I = \sum_{\alpha \in A} I_\alpha$, then $V(I) = \bigcap_{\alpha \in A} V(I_\alpha)$. Using this and the fact that principal open sets are a base for the Zariski topology on the $U_i$'s, you reduce to the special case that $U = \phi_i^{-1}(D(f))$ for some $i$ and $f \in k[U_i]$. So $f$ is a polynomial in $T_0/T_i, \ldots, T_n/T_i$.

Scaling by a large power of $T_i$, we obtain a polynomial $f^* \in S$ such that $f^*$ is zero outside $U_i$ and

$$f^*(x) \neq 0 \text{ if and only if } f(x) \neq 0$$

for $x \in U_i$. Hence, $U$ is the complement of $V(f^*)$. \hfill $\square$

**Exercise 1.23.**

(4) Show that all affine varieties are quasi-projective varieties.

(5) Define a map of sets $\phi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m}$ by

$$\phi([x_0, \ldots, x_n], [y_0, \ldots, y_m]) = [x_iy_j]_{0 \leq i \leq n, 0 \leq j \leq m}.$$
Show that the image of $\phi$ is a closed subset of $\mathbb{P}^{mn+m+n}$, and that $\phi$ is an isomorphism of varieties between $\mathbb{P}^n \times \mathbb{P}^m$ and its image. Hence: products of projective varieties are projective.

1.8. Dimension. Let $X$ be an irreducible variety. If $X$ is affine, its coordinate ring $k[X]$ is then an integral domain. So we can form its field of fractions, denoted $k(X)$. Now let $0 \neq f \in k[X]$, and consider the affine open subvariety $D(f)$ of $X$. Its coordinate ring is $k[D(f)] = k[X]_f$, and its field of fractions $k(D(f))$ is canonically isomorphic to $k(X)$. Finally, let $U$ be any non-empty affine open subset of $X$. Pick $f$ such that $D(f) \subseteq U$. Then, by the preceding argument applied to $U$, $k(U)$ (the field of fractions of the coordinate ring of $U$) is canonically isomorphic to $k(D(f)) \cong k(X)$. We have shown: all affine open subsets of an irreducible affine variety have canonically isomorphic fields of fractions.

Now let $X$ be an arbitrary irreducible variety. Define $k(X)$ to be the field of fractions of any non-empty affine open subset $U$ of $X$. By the previous paragraph, if $V$ is some other non-empty affine open subset of $X$, then $k(U) \cong k(U \cap V) \cong k(V)$ (canonically). So $k(X)$ is well-defined up to canonical isomorphism. It is an important invariant of $X$, called the function field of $X$.

Now define the dimension of $X$ to be the transcendence degree of the field extension $k(X) : k$. For example, $k(\mathbb{P}^n) \cong k(\mathbb{A}^n) \cong k(T_1, \ldots, T_n)$. So $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$. A good rule: if your variety $X$ is not irreducible, you should not be talking about its dimension (but you can talk about the dimensions of its irreducible components).

We only need for now a couple of lemmas about dimension:

Lemma 1.24. If $X$ is irreducible and $Y$ is a proper irreducible closed subset, then $\dim Y < \dim X$.

Proof. Let $U$ be an affine open subset of $X$ which has non-empty intersection with $Y$. Then, $\dim X = \dim U$ and $\dim Y = \dim (U \cap Y)$. In other words, replacing $X$ by $U$ and $Y$ by $U \cap Y$, we may assume $X$ and $Y$ are both affine.

Let $A = k[X]$ and $P = I(Y)$, a non-zero prime ideal of $X$. Suppose that $\dim Y = e$, $\dim X = d$. Let $y_1, \ldots, y_e$ are algebraically independent elements of $k[Y] = A/P$. Then their pre-images $x_1, \ldots, x_e$ in $A$ are also algebraically independent, hence $e \leq d$. Now assume $e = d$. Let $f$ be a non-zero element of $P$. There is a relation $p(f, x_1, \ldots, x_e) = 0$ where $p$ is a polynomial in $e + 1$ variables $T_0, T_1, \ldots, T_e$ (because $f, x_1, \ldots, x_e$ are dependent). WMA by dividing through by $T_0$ if necessary that $p$ is not divisible by $T_0$. But then the polynomial $q := p(0, T_1, \ldots, T_e) \in k[T_1, \ldots, T_e]$ is non-zero. But $q(y_1, \ldots, y_e) = 0$ so $q$ is a dependency between the $y$’s, which is a contradiction. Hence $e < d$. \qed

Lemma 1.25. If $X$ and $Y$ are irreducible varieties, then $X \times Y$ is again an irreducible variety and $\dim X \times Y = \dim X + \dim Y$. 
Proof. Let $X = U_1 \cup \cdots \cup U_n, Y = V_1 \cup \cdots \cup V_m$ be affine open covers. Each $U_i \times V_j$ is irreducible affine, since each $k[U_i] \otimes k[V_j]$ is an integral domain. Now $(U_1 \times V_1) \cap (U_i \times V_j)$ is a non-empty open subset of $U_i \times V_j$, hence is dense in $U_i \times V_j$. This shows that $U_1 \times V_1 \supseteq U_i \times V_j$. Hence, $U_1 \times V_1 = X \times Y$. Now suppose $X \times Y = Z_1 \cup Z_2$ for closed sets $Z_1, Z_2$. By irreducibility of $U_1 \times V_1$, we must have $Z_i \supseteq U_1 \times V_1$ for some $i$. But then $Z_i = U_1 \times V_1 = X \times Y$, so $X \times Y$ is irreducible.

Now to compute $\dim(X \times Y)$, it equals $\dim(U_1 \times V_1)$. Let $x_1, \ldots, x_d$ and $y_1, \ldots, y_e$ be maximal sets of algebraically independent elements of $k[U_1], k[V_1]$ respectively. Then, $x_1 \otimes 1, \ldots, x_d \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_e$ is a maximal set of algebraically independent elements of $k[U_1] \otimes k[V_1]$. Hence, $\dim(X \times Y) = \text{tr.deg.}(k[U_1] \times V_1) = d + e = \dim U_1 + \dim V_1 = \dim X + \dim Y$.

Exercise 1.26. (6) Give an example of an affine variety having two irreducible components, one of dimension 1, the other of dimension 2.

1.9. A theorem about morphisms. We’ll slowly need to know more about algebraic geometry, but I’ll try to introduce it as we go along from now on. But I need one theorem right now about morphisms of varieties, which can be very subtle beasts in general.

Let me call a morphism $\phi : X \to Y$ dominant if its image $\phi X$ is a dense subset of $Y$.

Lemma 1.27. Let $\phi : X \to Y$ be a morphism of affine varieties, and let $\phi^* : k[Y] \to k[X]$ be its comorphism.

(i) If $\phi^*$ is surjective, then $\phi X$ is a closed subset of $Y$.

(ii) $\phi^*$ is injective if and only if $\phi$ is dominant.

(iii) If $X$ is irreducible, so is $\overline{\phi X}$ and $\dim \overline{\phi X} \leq \dim X$.

Proof. (i) The statement that $\phi(x) = y$ is equivalent to saying $ev_y = ev_x \circ \phi^*$. Hence, $M_y = \ker ev_y = \ker (ev_x \circ \phi^*) \supseteq (\phi^*)^{-1}(M_x)$. In general, equality need not hold here: the inverse image of a maximal ideal under an algebra homomorphism need not be a maximal ideal!!! However if $\phi^*$ is surjective, that is true and we see that $\phi(x) = y$ if and only if $M_y = (\phi^*)^{-1}(M_x)$. Identifying $X$ now with the maximal ideals of $k[X]$ and $Y$ with the maximal ideals of $k[Y]$ (see 1.8), we see that $\phi X$ is the set of all maximal ideals of $k[Y]$ that are inverse images of maximal ideals of $k[X]$, i.e. by the correspondence theorem they are the maximal ideals of $k[Y]$ that contain $I = \ker \phi^*$. In other words, $\phi X = V(I)$ which is closed.

(ii) Note

$$I(\phi X) = \{ f \in k[Y] \mid f(\phi(x)) = 0 \text{ for all } x \in X \}$$

$$= \{ f \in k[Y] \mid \phi^* f = 0 \} = \ker \phi^*.$$
Hence, $\phi^*$ is injective if and only if $I(\phi X) = \{0\}$, which is if and only if $V(I(\phi X)) = \overline{\phi X} = Y$.

(iii) We already know the statement about irreducibility, see 1.4. Now consider the restriction $\psi : X \to \overline{\phi X}$ of $\phi$. It is dominant, so by (ii) $\psi^* : k[\overline{\phi X}] \to k[\phi X]$ is injective. This obviously implies $\dim \overline{\phi X} \leq \dim X$ (by the definition of dimension as the maximal number of algebraically independent elements).

\textbf{Theorem 1.28.} Let $\phi : X \to Y$ be a morphism of varieties. Then $\phi X$ contains a non-empty open subset of its closure $\overline{\phi X}$.

\textit{Proof.} Let us make some reductions:

(1) We can find an affine open $V$ of $Y$ such that $\phi^{-1}V$ is non-empty. Let $U$ be an affine open subset of $\phi^{-1}V$. It then suffices to show that $\phi U$ contains a non-empty open subset of its closure. In other words, replacing $X$ by $U$, $Y$ by $V$ and $\phi$ by its restriction, we may assume $X$ and $Y$ are affine.

(2) Let $X_1, \ldots, X_s$ be the irreducible components of $X$. Then $\overline{\phi X} = \overline{\phi X_1} \cup \cdots \cup \overline{\phi X_s}$. Hence, we can replace $X$ by one of the $X_i$ to assume $X$ is irreducible.

(3) Finally replace $Y$ by $\overline{\phi X}$ to assume that $\phi$ is dominant.

Now we have reduced to the following situation: $\phi : X \to Y$ is a morphism of affine varieties, $\phi^* : k[Y] \to k[\phi X]$ is injective, and $k[\phi X]$ is an integral domain.

Now apply the following technical lemma from commutative algebra to $B = k[X], A = k[Y]$ (see Springer 1.9.4 for a proof):

Let $A \subset B$ be finitely generated algebras, with $B$ an integral domain. Then there exists $0 \neq a \in A$ such that any homomorphism $f : A \to k$ with $f(a) \neq 0$ can be extended to a homomorphism $\tilde{f} : B \to k$.

We get some element $0 \neq a \in k[Y]$ with the property that for each of the evaluation homomorphisms $ev_y$ for $y \in Y$ satisfying $ev_y(a) = a(y) \neq 0$, there exists $x \in X$ with $ev_y = ev_x \circ \phi^*$. But that is exactly saying that each $y \in D(a)$ is $\phi(x)$ for some $x \in X$, i.e. $D(a) \subseteq \phi X$. We are done. \hfill \Box

\textbf{Exercise 1.29.} (7) Give an example of a morphism $\phi : X \to Y$ of irreducible affine varieties such that $\phi^* : k[Y] \to k[\phi X]$ is injective but $\phi$ is not surjective.