ALGEBRAIC GROUPS

Disclaimer: There are millions of errors in these notes!

1. Some algebraic geometry

The subject of algebraic groups depends on the interaction between algebraic geometry and group theory. To get going, we therefore need to learn some basic algebraic geometry. I'll try to keep it to a minimum. Note I am following the book by Springer very closely here...

1.1. **Algebraic sets.** Let k be an algebraically closed field ALWAYS. We can think of the elements of the polynomial algebra

$$S = k[T_1, T_2, \dots, T_n] (= k[T] \text{ for short})$$

as k-valued functions on the space k^n: if $v = (v_1, \dots, v_n) \in k^n$ and $f = f(T_1, \dots, T_n)$ then $f(v) = f(v_1, \dots, v_n)$. We call $v \in k^n$ a *zero* of f if $f(v) = 0$.

Given an ideal $I \subset S$, let $V(I)$ denote the set of all common zeros of all functions in I, i.e.

$$V(I) = \{v \in k^n \mid f(v) = 0 \text{ for all } f \in I\}.$$

Given a subset $X \subset k^n$, let $I(X)$ denote the ideal of all functions $f \in S$ vanishing on all of X, i.e.

$$I(X) = \{f \in S \mid f(v) = 0 \text{ for all } v \in X\}.$$

Note that

$$X \subseteq V(I(X)), \qquad I \subseteq I(V(I)).$$

But neither one need be equality. To understand the problem, recall the *radical* $\sqrt{I}$ of an ideal I is the ideal

$$\{f \in S \mid f^n \in I \text{ for some } n \geq 1\}.$$

It is obvious that the ideal $I(X)$ is a radical ideal, i.e. equals its radical. Now we need Hilbert's Nullstellensatz:

Nullstellensatz.

(i) *If I is a proper ideal of S (i.e. $I \neq S$) then $V(I)$ is non-empty.*
(ii) *For any ideal I of S, $I(V(I)) = \sqrt{I}$.*

By an *algebraic set* in k^n we mean one of the form $V(I)$ for some ideal I of S. Using the Nullstellensatz, you check that the operators V and I set up a 1–1 inclusion reversing correspondence between the *radical ideals* of S and the *algebraic sets* in k^n.

The function $I \mapsto V(I)$ has the following properties:

(a) $V(\{0\}) = k^n, V(S) = \emptyset$;

(b) $V(I \cap J) = V(I) \cup V(J)$;
(c) If $(I_\alpha)_{\alpha \in A}$ is a family of ideals and $I = \sum_{\alpha \in A} I_\alpha$, then $V(I) = \bigcap_{\alpha \in A} V(I_\alpha)$.

(These are all easy to prove except perhaps for (b). For this, take $x \in V(I) \cup V(J)$, wlog $x \in V(I)$. Then everything in I vanishes on x. So everything in $I \cap J$ certainly vanishes on x, i.e. $x \in V(I \cap J)$. Conversely, take $x \notin V(I) \cup V(J)$. Then we can find some functions $f \in I$ and $g \in J$ such that $f(x) \neq 0, g(x) \neq 0$. Then $(fg)(x) \neq 0$. But $fg \in I \cap J$ hence $x \notin V(I \cap J)$.) Hence:

Lemma 1.1. *The algebraic sets in k^n form the* closed sets *of a topology on k^n called the* Zariski topology*.*

Example 1.2.
(i) The closed sets in k, other than k itself, are the finite ones.
(ii) Given $X \subseteq k^n$, let $\overline{X}$ denote its closure in the Zariski topology. Then, $\overline{X} = V(I(X))$.
(iii) On $\mathbb{C}^n$, the closed sets in the Zariski topology are closed sets in the Euclidean topology. But there are MANY closed sets in the Euclidean topology that are not Zariski closed. The Zariki topology has *very few* closed sets, for example it is not Hausdorff.

By the way, the points in k^n are closed: e.g. $v = (v_1, \dots, v_n)$ is the zero set of the maximal ideal of S generated by $T_1 - v_1, \dots, T_n - v_n$. Indeed, by the Nullstellensatz the points of k^n parametrize the maximal ideals in S in this way.

Now recall by the Hilbert basis theorem that S is a Noetherian ring, i.e. it has ACC on ideals or equivalently any nonempty collection of ideals of S has a maximal element relative to inclusion. Hence:

(a) k^n has DCC on closed sets;
(b) Any non-empty family of closed sets in k^n has a minimal one.

Note a topological space satisfying either of the equivalent properties (a) or (b) is called a *Noetherian topological space.*

Lemma 1.3. *k^n is quasi-compact, i.e. every open cover has a finite open subcover. (I reserve the term "compact" for Hausdorff spaces, which k^n is not.)*

Proof. In terms of closed sets this says that if $(I_\alpha)_{\alpha \in A}$ is a family of ideals of S such that $\bigcap_{\alpha \in A} V(I_\alpha) = \emptyset$, there is a finite subset A_0 of A such that $\bigcap_{\alpha \in A_0} V(I_\alpha) = \emptyset$. We know $V(\sum_{\alpha \in A} I_\alpha) = \emptyset$. Hence, by the Nullstellensatz, $\sum_{\alpha \in A} I_\alpha = S$. Hence there are finitely many of the I_α such that 1 lies in their sum, i.e. $\sum_{\alpha \in A_0} I_\alpha = S$ for some finite subset $A_0 \subset A$. But then $\bigcap_{\alpha \in A_0} V(I_\alpha) = V(S) = \emptyset$ and we are done. □

Finally, note that if X is any algebraic set in k^n, with the subspace topology, it is also *Noetherian* and *quasi-compact.*

1.2. **Irreducible topological spaces.** Let X be a non-empty topological space. Then, X is called *reducible* if it is the union of two proper closed subsets. Otherwise, X is called *irreducible*. A subset $A \subset X$ is called irreducible if it is irreducible in the induced topology.

Exercises 1.4. Let X be a topological space.

(1) $A \subset X$ is irreducible if and only if $\overline{A}$ is.
(2) Let $f : X \to Y$ be a continuous map of topological spaces. If X is irreducible, then so is $f(X)$.

Lemma 1.5. *Let X be a Noetherian topological space. Then X has finitely many maximal irreducible subsets. Moreover, these are closed and cover X.*

Proof. By 1.4, maximal irreducible subsets are closed.

Now suppose that X cannot be written as a union of finitely many irreducible closed subsets. Using the Noetherian property, we can find a minimal non-empty closed subset A of X which is not a finite union of irreducible closed subsets. Clearly, A is reducible, so it is a union of two proper closed subsets. But by the minimality of A each of these is a finite union of irreducible closed subsets, hence A is too, a contradiction.

Therefore we can write $X = X_1 \cup \cdots \cup X_s$ where the X_i are irreducible and closed. We may assume that there are no inclusions amongst them. Now suppose that Y is an irreducible subset of X. Then, $Y = (Y \cap X_1) \cup \cdots \cup (Y \cap X_s)$, so by definition of irreducibility we get that $Y \subseteq X_i$ for some i. Hence, any irreducible subset of X is contained in one of the X_i. Hence, the X_i are the maximal irreducible subsets of X. □

The maximal irreducible subsets of X are called the *irreducible components* of X.

We want to apply this theory to study algebraic sets in k^n, recalling that the Zariski topology is Noetherian. What does irreducibility mean in this context?

Lemma 1.6. *A closed subset X of k^n is irreducible if and only if $I(X)$ is a prime ideal.*

Proof. Let X be irreducible. To show $I(X)$ is a prime ideal, suppose $f, g \in S$ and $fg \in I(X)$. Then, $X \subseteq V(f) \cup V(g)$, i.e.

$$X = (X \cap V(f)) \cup (X \cap V(g)).$$

Since X is irreducible, we must therefore have wlog that $X \subseteq V(f)$. So $f \in I(X)$, and $I(X)$ is prime.

Conversely, suppose $I(X)$ is a prime ideal but that X is reducible. Suppose $X = V(I) \cup V(J) = V(I \cap J)$ with $V(I) \subsetneq X$, for radical ideals I, J. Then $I(X) \subsetneq I$ so we can pick $f \in I - I(X)$. But $fg \in I(X)$ for all $g \in J$, hence by primeness, $g \in I(X)$. This shows that $J \subseteq I(X)$, whence $X \subseteq V(J)$, and X is irreducible. □

So now we know that any closed set X in k^n can be decomposed uniquely into irreducible components as

$$X = X_1 \cup \cdots \cup X_m.$$

Moreover, each $X_i = V(P_i)$ for some prime ideal in S. You can rephrase this purely in terms of algebra if you like: any radical ideal I of S is an intersection of prime ideals. Moreover, assuming there are no inclusions amongst them, these prime ideals are uniquely determined.

Exercises 1.7. (3) Describe all irreducible Hausdorff topological spaces.

(4) Let $X = \{(x, y) \in k^2 \mid xy = 0\}$. Show that X is a closed, connected subset of k^2. What are its irreducible components?

(5) Show that the Zariski topology on k^2 is NOT the same as the product topology on $k \times k$ arising from the Zariski topology on each copy of k.

1.3. **Affine algebras.** So far we can consider the Zariski topology on a closed subset X of k^n. We would like a more intrinsic description of the topology on X that does not depend on its embedding in k^n.

To start with, let X be a closed set in k^n. Let

$$k[X] = S/I(X),$$

called the *coordinate ring* of X. Clearly elements of $k[X]$ can be viewed as functions on X in a well-defined way. Indeed, restriction of functions from k^n to X determines a natural surjection $S \to k[X]$ with kernel $I(X)$. Now, $k[X]$ has the following properties:

(a) $k[X]$ is finitely generated (as a k-algebra);
(b) $k[X]$ is reduced (0 is its only nilpotent element).

We will call a k-algebra with these properties an *affine algebra*.

Clearly every affine algebra arises as the coordinate ring of some algebraic set X. Moreover, we can completely recover the topological space X from its coordinate ring:

Lemma 1.8. *Let X be an algebraic set with coordinate ring $k[X]$.*

(i) *For each point $x \in X$, let M_x denote the set of all $f \in k[X]$ vanishing at x. The map $x \mapsto M_x$ is a bijection between the points of X and the set* $\max(k[X])$ *of maximal ideals of $k[X]$.*
(ii) *For each ideal I of $k[X]$, let $V(I)$ denote the set of all common zeros of all functions in I. Then, the closed sets of X are the $V(I)$ as I runs over all ideals of $k[X]$.*

Proof. Everything follows because the ideals of $k[X]$ are in 1–1 correspondence with the maximal ideals of S containing $I(X)$. □

The point of the lemma is that you can completely recover X and its Zariski topology from the algebra $k[X]$. By the way, X is irreducible if and only if its coordinate ring $k[X]$ is an integral domain, by 1.6.

So now let X be an algebraic set with coordinate ring $k[X]$. (Note I no longer need to talk about the embedding $X \hookrightarrow k^n$!). Let us now introduce some special *open* subsets of X: for $f \in k[X]$ let

$$D(f) = \{x \in X \mid f(x) \neq 0\} = X - V(f).$$

By the way,

$$D(fg) = D(f) \cap D(g), \qquad D(f^n) = D(f).$$

The $D(f)$ are called *principal open subsets* of X.

Lemma 1.9. (i) *If $f, g \in k[X]$ and $D(f) \subseteq D(g)$ then $f^n \in (g)$ for some $n \geq 1$.*
(ii) *The principal open sets form a base for the topology on X.*

Proof. (i) Both statements are equivalent to $\sqrt{(f)} \subseteq \sqrt{(g)}$.
(ii) Equivalently, every closed set is an intersection of $V(f)$'s. But $V(I) = V(f_1) \cap \dots \cap V(f_n)$ if $I = (f_1, \dots, f_n)$, so this is obvious. □

1.4. **Affine algebraic varieties.** To start with let X be an algebraic set with coordinate ring $k[X]$. We understand the "polynomials" in $k[X]$ which make sense as functions defined on all of X. But if we are only interested in functions defined on a principal open set $D(f)$, more functions make sense: any "rational function" of the form g/f^n makes sense as a function on $D(f)$ since $f \neq 0$ there. In other words, elements of the *localization* $k[X]_f$ of $k[X]$ at the multiplicative set $\{1, f, f^2, \dots\}$ make sense as functions on $D(f)$.

Let us try to make this more precise: we want to work out exactly which functions make sense on an open subset U of X.

(R1) Start with a point $x \in X$. A k-valued function f defined in a neighbourhood of x is called *regular at x* if there are $g, h \in k[X]$ and an open neighbourhood $V \subseteq U \cap D(h)$ such that $f(y) = g(y)/h(y)$ for all $y \in X$. So: a regular function at x is one that looks like g/h near x.

(R2) Now let U be a non-empty open subset of X. A k-valued function f defined on U is called *regular* if it is regular at all points of U. We denote by $\mathcal{O}_X(U)$ or $\mathcal{O}(U)$ the algebra of all regular functions on U.

Notice the following properties are obvious:

(A) If U and V are non-empty open subsets of X with $U \subset V$, restriction defines a k-algebra homomorphism $\mathcal{O}(V) \to \mathcal{O}(U)$.

(B) Let $U = \bigcup_{\alpha \in A} U_\alpha$ be an open covering of an open set U. Suppose we are given $f_\alpha \in \mathcal{O}(U_\alpha)$ such that whenever $U_\alpha \cap U_\beta$ is non-empty, f_α and f_β restrict to the same element of $\mathcal{O}(U_\alpha \cap U_\beta)$. Then there is $f \in \mathcal{O}(U)$ whose restriction to each U_α is equal to f_α.

We have now constructed an example $\mathcal{O}_X$ of a *sheaf of functions* on the topological space X. You can probably guess the general definition: for an arbitrary topological space X, a sheaf $\mathcal{O}_X$ of functions on X determines for each open subset U of X a subalgebra $\mathcal{O}_X(U)$ of the algebra of all k-valued functions on U, such that the properties (A) and (B) above hold. The pair $(X, \mathcal{O}_X)$ is then called a *ringed space*. Let me record a few other generalities to do with ringed spaces:

(C) Let $(X, \mathcal{O}_X)$ be a ringed space and let U be an open subset of X. Then we have a new ringed space $(U, \mathcal{O}_X|_U)$ where U is given the subspace topology and $\mathcal{O}_X|_U$ is the *restriction of* $\mathcal{O}_X$ *to* U, defined on an open $V \subseteq U$ simply by $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$.)

(D) Let $(X, \mathcal{O}_X)$ be a ringed space and let $x \in X$ be a point. We denote by $\mathcal{O}_{X,x}$ or $\mathcal{O}_x$ the algebra

$$\varinjlim_{U \ni x} \mathcal{O}_X(U).$$

Thus, elements of $\mathcal{O}_x$ consist of equivalence classes of pairs (U, f) where U is an open neighbourhood of x and $f \in \mathcal{O}_X(U)$, two such pairs (U, f) and (V, g) being equivalent if there is an open neighbourhood $W \subseteq U \cap V$ such that $f|_W = g|_W$.

(E) Finally let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. Let $\phi : X \to Y$ be a continuous map. For each open set $V \subseteq Y$ and $f \in \mathcal{O}_Y(V)$, we can consider $\phi^* f = f \circ \phi$ as a function on $\phi^{-1}V$. This need not belong to $\mathcal{O}_X(\phi^{-1}V)$. If it does for all V and f, then ϕ is called a *morphism of ringed spaces*. So: a morphism of ringed spaces is a continuous map $\phi : X \to Y$ such that $\phi^*(\mathcal{O}_Y(V)) \subseteq \mathcal{O}_X(\phi^{-1}V)$ for all open subsets V of Y.

Now for an important definition: an *affine algebraic variety* is a ringed space $(X, \mathcal{O}_X)$ that is isomorphic (as a ringed space) to an algebraic set equipped with its sheaf of regular functions, as defined in (R2) above. In particular, the ringed space arising from the algebraic set $X = k^n$ itself is the affine algebraic variety denoted $\mathbb{A}^n$, *affine* n*-space*.

You probably think the definition of affine variety is a little stupid. After all, the pair $(X, \mathcal{O}_X)$ was built from the pair $(X, k[X])$. In turn, X was built from $k[X]$ as the set of all maximal ideals. So: there is *no more information* in the ringed space $(X, \mathcal{O}_X)$ than there was in the original affine algebra $k[X]$ we started with, though the point of view is very different. Actually, there might be *much less* information in $(X, \mathcal{O}_X)$ than there was in $k[X]$. That is the point of the next theorem: we can recover $k[X]$ out of the sheaf $\mathcal{O}_X$.

Let X be an algebraic set with coordinate ring $k[X]$, and let $(X, \mathcal{O}_X)$ be the corresponding affine algebraic variety. By the definitions, there is a natural map $\phi : k[X] \to \mathcal{O}_X(X)$.

Theorem 1.10. *ϕ is an isomorphism.*

Proof. By the Nullstellensatz, the only function in $k[X]$ which is zero on all of X is 0. This shows that ϕ is injective. The problem is surjectivity...

Let $f \in \mathcal{O}_X(X)$. For each $x \in X$ we can find an open neighbourhood U_x of x and $g_x, h_x \in k[X]$ such that h_x is non-zero on all of U_x and for each

$y \in U_x$,

$$f(y) = g_x(y)/h_x(y).$$

This is just the definition of the functions in $\mathcal{O}_X(X)$. Since the principal open sets form a base for the Zariski topology, we may assume $U_x = D(a_x)$ for some $a_x \in k[X]$. So $D(a_x) \subseteq D(h_x)$, hence there exists $h'_x \in k[X]$ and an integer $n_x \geq 1$ with

$$a_x^{n_x} = h_x h'_x.$$

Note $f = g_x h'_x/(a_x^{n_x})$ on U_x. So since $D(a_x) = D(a_x^{n_x})$ we may as well replace g_x by $g_x h'_x$ and h_x by $a_x^{n_x}$. In other words, we may assume that $U_x = D(h_x)$.

Now, X is quasi-compact so finitely many of the $D(h_x)$ cover X. Say $X = D(h_1) \cup \cdots \cup D(h_n)$. Now we have picked $h_1, \ldots, h_n$ and $g_i \in k[X]$ such that the restriction of f to $D(h_i)$ equals $g_i h_i^{-1}$.

Now, $g_i h_i^{-1}$ and $g_j h_j^{-1}$ coincide on $D(h_i) \cap D(h_j)$, and $h_i h_j$ vanishes outside of this set. So $h_i h_j (g_i h_j - g_j h_i) = 0$. Since the $D(h_i)$ cover X, the ideal generated by $h_1^2, \ldots, h_n^2$ is $k[X]$. So there exist $b_i \in k[X]$ with

$$1 = \sum_{i=1}^{n} b_i h_i^2.$$

Finally, let $x \in D(h_j)$. Then,

$$\begin{aligned} h_j(x)^2 \sum_{i=1}^{n} b_i(x) g_i(x) h_i(x) &= \sum_{i=1}^{n} b_i(x) h_i(x)^2 h_j(x) g_j(x) \\ &= h_j(x) g_j(x) = h_j(x)^2 f(x). \end{aligned}$$

This shows that $f = \phi(\sum_{i=1}^{n} b_i g_i h_i)$ giving surjectivity. □

Exercise 1.11. (6) Let $D(f)$ be a principal open subset of X. Copy the proof of the above theorem to show that there is a natural isomorphism $k[X]_f \to \mathcal{O}_X(D(f))$. (Remember $k[X]_f$ denotes the localization of $k[X]$ at $\{1, f, f^2, \ldots\}$.)

(7) Let $(X, \mathcal{O}_X)$ be an affine algebraic variety. For a point $x \in X$, we have defined (D) the algebra $\mathcal{O}_x$ of regular functions at x. Let $M_x = \{f \in k[X] \mid f(x) = 0\}$. Prove that $\mathcal{O}_x$ is isomorphic to the localization $k[X]_{M_x}$ of $k[X]$ at the multiplicative set $k[X] - M_x$. Hence: $\mathcal{O}_x$ is a *local ring* with unique maximal ideal $\mathfrak{m}_x = \{(U, f) \mid f(x) = 0\}$.

We have defined in (E) the notion of morphism of ringed space. Now let X and Y be affine varieties. Then, according to the definition (E), we should call a map $\phi : X \to Y$ a *morphism of varieties* if it is a continuous map such that $\phi^*(\mathcal{O}_Y(V)) \subseteq \mathcal{O}_X(\phi^{-1}V)$ for each open subset V of Y. In particular this means by 1.10 that ϕ defines a map $\phi^* : k[Y] \to k[X]$ of the associated coordinate rings, called the *comorphism* of ϕ.

Conversely, suppose we are just given a map $\psi : k[Y] \to k[X]$ of algebras. For $x \in X$, let ev_x be the homomorphism $k[X] \to k$ determined by evaluation at x. Then there is a unique map $\phi : X \to Y$ such that $ev_{\phi(x)} = ev_x \circ \psi$ for all $x \in X$ (see 1.8(i)). You should check for yourself that this map is a *morphism of ringed spaces*; it is obvious that $\phi^* = \psi$.

Exercise 1.12. (8) Show that a morphism $\phi : X \to Y$ of affine varieties is an isomorphism (i.e. has a two-sided inverse) if and only if $\phi^* : k[Y] \to k[X]$ is an isomorphism of algebras.

(9) Suppose that k is of characteristic $p > 0$. Consider the map $\phi : \mathbb{A}^n \to \mathbb{A}^n, (v_1, \dots, v_n) \mapsto (v_1^p, \dots, v_n^p)$. Show that ϕ is a bijective morphism of affine varieties, but is *not* an isomorphism of affine varieties.

(10) Let X be an affine algebraic variety and $f \in k[X]$. Show that $(D(f), \mathcal{O}_X|_{D(f)})$ is again an affine algebraic variety, with coordinate ring $k[X]_f$.

Now let X and Y be two affine varieties. So X and Y are topological spaces equipped with sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ of regular functions. We also have their coordinate rings $k[X] = \mathcal{O}_X(X)$ and $k[Y] = \mathcal{O}_X(Y)$. We understand the notion of a morphism $\phi : X \to Y$: a continuous map such that $\phi^*(\mathcal{O}_Y(V)) \subseteq \mathcal{O}_X(\phi^{-1}V)$ for every open in V. But we have shown that these properties are entirely equivalent to saying simply that the comorphism ϕ^* associated to the function ϕ maps $k[Y]$ into $k[X]$.

Example 1.13. Let $X = \mathbb{A}^3$ and $Y = V(T_1T_2 - 1) \subset \mathbb{A}^2$. So $k[X] = k[T_1, T_2, T_3]$ and $k[Y] = k[T_1, T_2]/(T_1T_2 - 1) = k[S_1, S_2]$ where S_i is the image of T_i.

(i) What do morphisms $\phi : X \to Y$ look like? Well, to make sense just as a function we must have that $\phi(v) = \phi(v_1, v_2, v_3) = (\phi_1(v), \phi_2(v))$ where $\phi_1(v)\phi_2(v) = 1$. Now what is the comorphism ϕ^* on S_i? Well, $\phi^*(S_i)(v_1, v_2, v_3) = S_i(\phi_1(v), \phi_2(v)) = \phi_i(v)$. Thus, ϕ_1 has to belong to $k[X]$, i.e. be a polynomial in T_1, T_2 and T_3. Similarly, ϕ_2 has to be a polynomial. Thus, $\phi(v) = (\phi_1(v), \phi_2(v))$ for *polynomials* ϕ_i. Moreover, $\phi_1\phi_2 = 1$, so they both have to be invertible polynomials, so $\phi_1 = c, \phi_2 = c^{-1}$ for a non-zero scalar c. So the only ϕ are the constant ones with $\phi(v) = (c, c^{-1})$, i.e. $\mathrm{Hom}(X, Y) \cong k^\times$.

(ii) What do morphisms $\phi : Y \to X$ look like? We must have $\phi(w) = \phi(w_1, w_2) = (\phi_1(w), \phi_2(w), \phi_3(w))$. What is $\phi^*(T_i)$? Its the function on Y with $\phi^*(T_i)(w) = \phi_i(w)$. So each coordinate function ϕ_i has to belong to $k[S_1, S_2]$. But that is all. The ϕ_i can be chosen arbitrarily in $k[S_1, S_2]$...

The moral: morphisms between affine varieties are functions whose coordinates are polynomials.