

MATH 681 TAKE HOME FINAL

Turn in by 5pm on Friday of finals week. (That is also the deadline to turn in HW3.) Feel free to look anything you like up in your notes as you work on these problems. It is up to you how long you spend on the exam. You might have better things to do – in which case limit yourself to TWO HOURS and don't worry if you can't do all the problems in that time. Just tell me how long you actually took...

1. Let X be the variety of all 2×2 matrices over \mathbb{C} whose square is zero.
 - (a) Show that

$$X = \left\{ \begin{bmatrix} x & b \\ c & -x \end{bmatrix} : x^2 + bc = 0 \right\}.$$

- (b) Substitute $b = y + z, c = y - z$ and sketch the resulting surface over \mathbb{R} .
 - (c) Prove carefully that the zero matrix is the only singular point. (Remark: for $n > 2$ the *nilpotent cone* of all $n \times n$ nilpotent matrices is a much more interesting singular variety.)
2. Let F be the flag variety consisting of nested chains $(V_0 \subset V_1 \subset V_2 \subset V_3)$ of subspaces of k^3 such that $\dim V_i = i$. Let

$$X = \{(V_0 \subset V_1 \subset V_2 \subset V_3) \in F \mid e_{1,2}V_i \subseteq V_{i-1} \text{ for } i = 1, 2, 3\}$$

where $e_{1,2}$ is the 3×3 matrix with a one in the 12-entry and zeros elsewhere. Describe the irreducible components of X explicitly. Is X connected?

(Remark: This X is the first example of a *Springer fiber*. More general Springer fibers arise by considering all flags in n -space annihilated by a fixed nilpotent matrix. The irreducible components and their intersections get quite interesting...)

3. Let G be a connected algebraic group and $x \in G$ be fixed.
 - (a) Define the *commutator morphism*

$$\gamma_x : G \rightarrow G, \quad y \mapsto yxy^{-1}x^{-1}.$$

Show that $(d\gamma_x)_e : \mathfrak{g} \rightarrow \mathfrak{g}$ is the map $X \mapsto X - (\text{Ad } x)X$.

- (b) Let

$$C_G(x) = \{g \in G \mid \text{Int } x(g) = g\},$$

$$\mathfrak{c}_\mathfrak{g}(x) = \{X \in \mathfrak{g} \mid \text{Ad } x(X) = X\}.$$

Show that the Lie algebra of $C_G(x)$ is contained in $\mathfrak{c}_\mathfrak{g}(x)$.

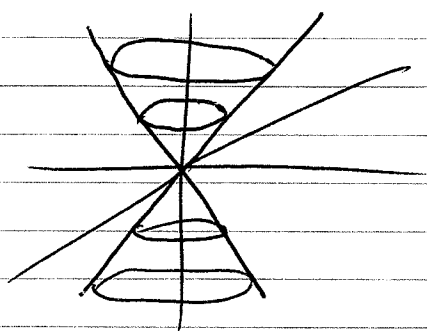
(Remark: in general equality need not hold here, but it always does in characteristic 0 and often does otherwise, for example it does if $G = GL_n(k)$ regardless of the ground field.)

Math 631 Take Home Final Solutions

1. (a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ squares to zero \Leftrightarrow characteristic poly is x^2
 $\Leftrightarrow \det = \text{trace} = 0$
 $\Leftrightarrow ad = bc, a+d=0$
 $\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x & b \\ c & -x \end{pmatrix}, x^2 + bc = 0$

(b) $b = y+z, c = y-z \Rightarrow \left\{ \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} \mid x^2 + y^2 = z^2 \right\}$

So $U_1 \cong \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 \}$



A cone!

(c) $k[X] = k[x, y, z] / (x^2 + y^2 - z^2) \quad \Omega_X = \frac{k[x] dx \otimes k[y] dy \otimes k[z] dz}{(2x dx + 2y dy - 2z dz)}$

($k = \mathbb{C}$ of course!)

\therefore Its tangent space is 2-d at all points where $2x dx + 2y dy - 2z dz$ evaluates to $\neq 0$, i.e. all points except $(0, 0, 0)$

\nearrow
 here tangent space is 3-d.

2. Let v_1, v_2, v_3 be standard basis.

Of course $V_0 = \{0\}$ $V_3 = k^3 = V$.

Need $e_{12}V_3 = \langle v_1 \rangle \subseteq V_2$

$\therefore V_2 = \langle v_1, av_2 + bv_3 \rangle$ some a, b not both zero.

Then $e_{12}V_2 = \langle av_1 \rangle \subseteq V_1$.

So If $a=0$ then $V_2 = \langle v_1, v_3 \rangle$ and V_1 is any 1-space in V_2

If $a \neq 0$ then $V_2 = \langle v_1, v_2 + \lambda v_3 \rangle$ and $V_1 = \langle v_1 \rangle$

Let $Y = \{ L \subset \langle v_1, v_3 \rangle \mid L \text{ any 1-space in } \langle v_1, v_3 \rangle \} \subset X$

Note Y is intersection of $V(T_2) \times \{ \langle v_1, v_3 \rangle \} \in G_1(V) \times G_2(V)$
with X

zero set of homog. polynomial so closed in $G_1(V)$

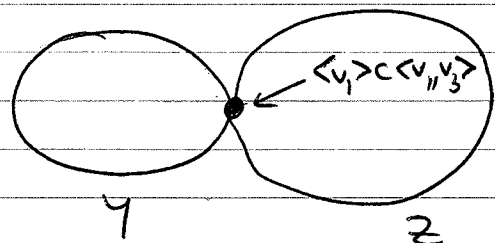
$\therefore Y$ is closed in X , obviously $Y \cong G_1(\langle v_1, v_3 \rangle) \cong \mathbb{P}^1$

Let $Z = \{ \langle v_1 \rangle \subset P \mid P \text{ any 2-space containing } \langle v_1 \rangle \} \subset X$

Note Z is intersection of $\{ \langle v_1 \rangle \} \times \{ P \in G_2(V) \mid P \supseteq \langle v_1 \rangle \}$

$\therefore Z$ is closed in X , obviously $Z \cong G_1(V/\langle v_1 \rangle) \cong \mathbb{P}^1$.

We've decomposed $X = Y \cup Z$
 $\uparrow \uparrow$
closed, $\cong \mathbb{P}^1$
 \therefore irreducible



Two copies of \mathbb{P}^1 glued at a point

3. (a) $\delta_x: G \rightarrow G, y \mapsto yxy^{-1}x^{-1}$

Consider composite $\delta_x: G \xrightarrow{(\text{id}, i)} G \times G \xrightarrow{(\text{id}, \text{Int}_x)} G \times G \xrightarrow{m} G$
 $y \mapsto (y, y^{-1}) \mapsto (y, xy^{-1}x^{-1}) \mapsto yxy^{-1}x^{-1}$

Known: $\left(\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{derivative} & \text{derivative} & \text{derivative} \\ \text{is } (\text{id}, -\text{id}) & \text{is } (\text{id}, \text{Ad}_x) & \text{is } p: (a, b) \mapsto a+b \end{array} \right)$

$\therefore (d\delta_x)_e(X) = p(X, -\text{Ad}_x(X)) = X - \text{Ad}_x(X)$

(b) $C_G(x) = \{g \in G \mid \delta_x(g) = e\} = \delta_x^{-1}(e)$ "fiber"

$I(C_G(x)) = \{f \in k[G] \mid f(\delta_x^{-1}(e)) = 0\}$

$\supseteq \delta_x^*(\ker \text{ev}_e)$

as if $f(e) = 0$ then $(\delta_x^* f)(g) = f(\delta_x(g)) = f(e) = 0$
and $g \in \delta_x^{-1}(e)$

$\therefore L(C_G(x)) = \{X \in \mathfrak{g} \mid X(I(C_G(x))) = 0\}$

$\subseteq \{X \in \mathfrak{g} \mid X(\delta_x^*(\ker \text{ev}_e)) = 0\}$

Important place!

$\ker \text{ev}_e \oplus k \cdot 1 = k[G]$
derivatives kill this always

$= \{X \in \mathfrak{g} \mid (d\delta_x)_e(X) = 0\}$

$= \{X \in \mathfrak{g} \mid (d\delta_x)_e X = 0\}$

$= \{X \in \mathfrak{g} \mid \text{Ad}_x(X) = X\}$ by (a) $= C_{\mathfrak{g}}(x)$