MATH 681 TAKE HOME FINAL

Turn in by 5pm on Friday of finals week. (That is also the deadline to turn in HW3.) Feel free to look anything you like up in your notes as you work on these problems. It is up to you how long you spend on the exam. You might have better things to do – in which case limit yourself to TWO HOURS and don't worry if you can't do all the problems in that time. Just tell me how long you actually took...

1. Let X be the variety of all 2×2 matrices over $\mathbb{C}$ whose square is zero.
   (a) Show that
   $$X = \left\{ \begin{bmatrix} x & b \\ c & -x \end{bmatrix} \,:\, x^2 + bc = 0 \right\}.$$
   (b) Substitute $b = y + z, c = y - z$ and sketch the resulting surface over $\mathbb{R}$.
   (c) Prove carefully that the zero matrix is the only singular point.

   (Remark: for $n > 2$ the *nilpotent cone* of all $n \times n$ nilpotent matrices is a much more interesting singular variety.)
2. Let F be the flag variety consisiting of nested chains $(V_0 \subset V_1 \subset V_2 \subset V_3)$ of subspaces of k^3 such that $\dim V_i = i$. Let
   $$X = \{(V_0 \subset V_1 \subset V_2 \subset V_3) \in F \mid e_{1,2} V_i \subseteq V_{i-1} \text{ for } i = 1, 2, 3\}$$
   where $e_{1,2}$ is the 3×3 matrix with a one in the 12-entry and zeros elsewhere. Describe the irreducible components of X explicitly. Is X connected?

   (Remark: This X is the first example of a *Springer fiber*. More general Springer fibers arise by considering all flags in n-space annihilated by a fixed nilpotent matrix. The irreducible components and their intersections get quite interesting...)
3. Let G be a connected algebraic group and $x \in G$ be fixed.
   (a) Define the *commutator morphism*
   $$\gamma_x : G \to G, \qquad y \mapsto yxy^{-1}x^{-1}.$$
   Show that $(d\gamma_x)_e : \mathfrak{g} \to \mathfrak{g}$ is the map $X \mapsto X - (\operatorname{Ad} x)X$.
   (b) Let
   $$C_G(x) = \{g \in G \mid \operatorname{Int} x(g) = g\},$$
   $$\mathfrak{c}_{\mathfrak{g}}(x) = \{X \in \mathfrak{g} \mid \operatorname{Ad} x(X) = X\}.$$
   Show that the Lie algebra of $C_G(x)$ is contain in $\mathfrak{c}_{\mathfrak{g}}(x)$.

   (Remark: in general equality need not hold here, but it always does in characteristic 0 and often does otherwise, for example it does if $G = GL_n(k)$ regardless of the ground field.)