Lie algebras – Examples sheet 4

Notation Always, $\mathfrak{g}$ is a semisimple Lie algebra. Fix a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, with corresponding root system Φ. Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be a base for Φ, Φ^+ the corresponding positive roots, W the Weyl group and $\epsilon : W \to \{\pm 1\}$ be the sign representation of W relative to the simple reflections $\{s_1, \dots, s_l\}$. Let $\omega_1, \dots, \omega_l$ denote the corresponding fundamental dominant weights, so that $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$. Let P and P^+ denote the integral and dominant integral weights respectively.

1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with standard basis e, f, h. Let τ be the endomorphism of $\mathfrak{g}$ defined by $\tau := \exp(\operatorname{ad} e)\exp(\operatorname{ad}(-f))\exp(\operatorname{ad} e)$. Verify explicitly that the automorphism τ acts on $\mathfrak{g}$ as conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Deduce that $\tau(e) = -f, \tau(f) = -e, \tau(h) = -h$.

2. If V and V' are finite dimensional $\mathfrak{g}$-modules, show that $\operatorname{ch}(V \oplus V') = \operatorname{ch}(V) + \operatorname{ch}(V')$ and $\operatorname{ch}(V \otimes V') = \operatorname{ch}(V).\operatorname{ch}(V')$.

3. Show that $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ can also be written as $\omega_1 + \cdots + \omega_l$.

4. For $\lambda \in P$, prove for yourself that $\operatorname{ch} M(\lambda) = \dfrac{e^\lambda}{\prod_{\alpha\in\Phi^+}(1 - e^{-\alpha})} = \dfrac{e^\lambda}{\sum_{w\in W} e^{w\rho-\rho}}$.

5. For the root system of type B_2, order the base so α_1 is the short root and α_2 is the long root. Compute the corresponding fundamental dominant weights $\{\omega_1, \omega_2\}$ in terms of α_1 and α_2, then write α_1 and α_2 in terms of ω_1 and ω_2. Do the same for A_2 and G_2 (hint: what are the Cartan matrices?).

6. Apply Weyl's character formula to compute the dimension of all weight spaces of the module $V(2\omega_1 + \omega_2)$ where $\mathfrak{g} = B_2$ and ω_1, ω_2 are as in question 5. Verify that the sum of the dimensions of all weight spaces equal the dimension as computed by Weyl's dimension formula.

7. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ with fundamental dominant weights ω_1, ω_2. Abbreviate $V(m_1\omega_1 + m_2\omega_2)$ by $V(m_1, m_2)$. Use Weyl's dimension formula to show

$$\dim V(m_1, m_2) = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2).$$

8. With notation as in question 7, show that $V(1,1) \otimes V(1,2) \cong V(2,3) \oplus V(3,1) \oplus V(0,4) \oplus V(1,2) \oplus V(1,2) \oplus V(2,0) \oplus V(0,1)$.

9. Use Weyl's dimension formula to show that a faithful, irreducible, finite dimensional $\mathfrak{g}$-module of smallest possible dimension has highest weight equal to ω_i for some i. Hence verify that the smallest dimension of a faithful, irreducible G_2-module is 7.

10. Labelling B_2's Dynkin diagram the way we usually do, explain *without making any calculations* why $\dim L(\omega_1) = 5$ and $\dim L(\omega_2) = 4$. (Hint for the second one: think about C_2).