Lie algebras – Examples sheet 3

1. Let \((a_{i,j})_{1 \leq i,j \leq l}\) be the Cartan matrix of an abstract root system. Prove directly from the definition that

(i) \(a_{i,i} = 2\);

(ii) \(a_{i,j} = 0\) if and only if \(a_{j,i} = 0\);

(iii) if \(i \neq j\) then \(a_{i,j} \leq 0\).

2. Prove that the only abstract root systems of rank two are

\[
\begin{align*}
A_1 & A_1 \\
\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
\end{align*}
\begin{align*}
A_2 & B_2 \\
\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}
\end{align*}
\begin{align*}
B_2 & G_2 \\
\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}
\end{align*}
\]

3. Let \(\Phi \subset E\) be an abstract root system with base \(\Delta\). For \(\alpha \in \Delta\), show that \(s_\alpha \in W\) stabilises \(\Phi^+ \setminus \{\alpha\}\). Deduce that \(s_\alpha(\rho) = \rho - \alpha\) for all \(\alpha \in \Delta\), where

\[
\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta.
\]

4. Let \(\Phi \subset E\) be an abstract root system.

(a) Show that

\[
\Phi' = \left\{ \frac{2\alpha}{(\alpha, \alpha)} \middle| \alpha \in \Phi \right\}
\]

is also an abstract root system in \(E\), known as the dual root system.

(b) Show that the Weyl group of \(\Phi'\) is isomorphic to the Weyl group of \(\Phi\).

(c) Show that \(\Phi'\) is irreducible if and only if \(\Phi\) is irreducible, and that the double dual of \(\Phi\) is isomorphic to \(\Phi\).

(d) Show that the dual root system to \(A_l, B_l, C_l\) or \(D_l\) is \(A_l, C_l, B_l\) or \(D_l\) respectively.

5. Let \(\Phi \subset E\) be an abstract root system.

(a) Let \(\Phi' \subset \Phi\) be a subset such that if \(\alpha_1, \ldots, \alpha_n \in \Phi'\) and \(\alpha = \sum a_i \alpha_i \in \Phi\) for certain coefficients \(a_i \in \mathbb{Z}\), then \(\alpha \in \Phi'\). Show that \(\Phi'\) is a root system in the subspace \(E' < E\) that it spans. Such subsystems of the root system \(\Phi\) are called closed subsystems.

(b) Verify that the set of long roots in the root system of type \(G_2\) is
a closed subsystem of type $A_2$, whereas the set of short roots in the root system of type $G_2$ is not a closed subsystem.

(c) More generally, show that the set of long roots in any irreducible root system is a closed subsystem.

(d) What subsystem does one obtain from the long roots in type $B_l$? Type $C_l$?

6. Let $\text{Aut} \Phi$ be the set of all automorphisms of the abstract root system $\Phi$, that is, all bijections $\theta : \Phi \to \Phi$ such that $\langle \theta(\alpha), \theta(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$.

(a) Show that $W$ is a normal subgroup of $\text{Aut} \Phi$.

(b) Let $\Gamma$ be the set of all $\theta \in \text{Aut} \Phi$ such that $\theta(\Delta)^+ = \Delta$, where $\Delta$ is a fixed base of $\Phi$. Show that $\text{Aut} \Phi$ is the semidirect product of $\Gamma$ and $W$, that is, $\text{Aut} \Phi = W\Gamma, W \cap \Gamma = 1$.

(c) Show that $\Gamma$ can be identified with the set of all automorphisms (of directed graphs) of the Dynkin diagram of $\Phi$.

(d) Prove that the map $\alpha \mapsto -\alpha$ ($\alpha \in \Phi$) is an automorphism of $\Phi$. For which irreducible root systems $\Phi$ is this map an element of the Weyl group $W$?

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