Lie algebras – Examples sheet 2

$\mathfrak{g}$ denotes a finite dimensional, semisimple Lie algebra over $\mathbb{C}$, unless otherwise stated.

Representations

1. If V and W are $\mathfrak{g}$-modules, we made $\hom_{\mathbb{C}}(V, W)$ into a $\mathfrak{g}$-module by setting $(x.f)(v) = x.f(v) - f(x.v)$ for all $x \in \mathfrak{g}, f \in \hom_{\mathbb{C}}(V, W), v \in V$. Verify directly that this gives a well-defined $\mathfrak{g}$-module structure on $\hom_{\mathbb{C}}(V, W)$.

2. Using the fact that the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ is simple, show that the Killing form $(x, y) := \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$ is related to the form $\langle x, y\rangle := \mathrm{tr}(xy)$ by $(x, y) = 2n\langle x, y\rangle$.

Representations of $\mathfrak{sl}_2(\mathbb{C})$

In these exercises, $L(m)$ denotes the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $m + 1$.

3. Embed $\mathfrak{sl}_2(\mathbb{C})$ into $\mathfrak{sl}_3(\mathbb{C})$ in its upper left hand 2×2 position. The restriction of the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$ defines an 8-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module V. Show that $V \cong L(0) \oplus L(1) \oplus L(1) \oplus L(2)$.

4. Let $V = L(1)$ denote the natural $\mathfrak{sl}_2(\mathbb{C})$-module, with its usual basis x_1, x_2. Let $P = \mathbb{C}[x_1, x_2]$ be the polynomial algebra in two variables, and extend the action of $\mathfrak{g}$ on $\mathbb{C}x_1 \oplus \mathbb{C}x_2 < P$ to all of P by the rule

$$z.fg = (z.f)g + f(z.g)$$

for all $z \in \mathfrak{g}, f, g \in P$. Show that this makes P into an infinite dimensional $\mathfrak{g}$-module, and that the subspace $P_m < P$ of homogeneous polynomials of degree m is a $\mathfrak{g}$-submodule of P. Show $P_m \cong L(m)$.

5. Let $0 \le m \le n$. Prove the *Clebsch-Gordan formula*:

$$L(m) \otimes L(n) \cong L(n - m) \oplus L(n - m + 2) \oplus \cdots \oplus L(n + m - 2) \oplus L(n + m).$$

The Jordan decomposition

6. Show that $\mathfrak{g}$ is nilpotent if and only if every element x of $\mathfrak{g}$ is nilpotent (ie $\mathrm{ad}_{\mathfrak{g}}\, x$ is a nilpotent endomorphism of $\mathfrak{g}$). Give an example to show that the statement "$\mathfrak{g}$ is semisimple if and only if every element x of $\mathfrak{g}$ is semisimple" (ie $\mathrm{ad}_{\mathfrak{g}}\, x$ is diagonalisable) is false.

7. Let $\mathfrak{g}$ be the Lie algebra $\mathbb{C}$ with multiplication $[xy] = 0$ for all $x, y \in \mathbb{C}$. Every element of $\mathfrak{g}$ is both semisimple and nilpotent. Verify that the following maps $\mathfrak{g} \to \mathfrak{gl}_2(\mathbb{C})$ are representations of $\mathfrak{g}$:

(a) $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$; (b) $x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$; (c) $x \mapsto \begin{pmatrix} x & x \\ 0 & x \end{pmatrix}$.

Show that in (a) every non-zero element of the image of $\mathfrak{g}$ is semisimple but not nilpotent and that in (b) every non-zero element of the image of $\mathfrak{g}$ is nilpotent but not semisimple. In (c), show that the semisimple and nilpotent parts of every non-zero element of the image of $\mathfrak{g}$ are not even elements of the image of $\mathfrak{g}$. Thus, the Jordan decomposition is false in general if $\mathfrak{g}$ is not a semisimple Lie algebra.

8. Let $\mathfrak{g}' < \mathfrak{g}$ be two semisimple Lie algebras. For $x \in \mathfrak{g}'$, show that its abstract Jordan decomposition regarded as an element of $\mathfrak{g}'$ agrees with its abstract Jordan decomposition regarded as an element of $\mathfrak{g}$.

The Cartan decomposition

9. Compute explicitly the Cartan decomposition of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ taking $\mathfrak{h}$ to be the set of all diagonal matrices (ie verify all the details from the lectures). This is the most important example on this sheet!!

10. Compute the restriction of the Killing form on $\mathfrak{sl}_n(\mathbb{C})$ to the set $\mathfrak{h}$ of all diagonal matrices directly (without using question 4). Hence verify directly that the restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.

11. Calculate explicitly the *Cartan integers* for $\mathfrak{sl}_n(\mathbb{C})$: the numbers $\frac{2(\alpha,\beta)}{(\beta,\beta)}$ for all α, β in the root system Φ (they should all be $0, 2$ or ± 1!).

12. If $\mathfrak{g}$ is semisimple, $\mathfrak{h}$ a maximal toral subalgebra, prove that $\mathfrak{h} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$.

13. Prove that every maximal toral subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ is one dimensional.

14. Prove that every three dimensional semisimple Lie algebra is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

15. Using just the Cartan decomposition, prove that no $4, 5$ or 7 dimensional semisimple Lie algebras exist.

Jon Brundan, 10/14/2005.