Lie algebras – Examples sheet 1

All Lie algebras and modules are finite dimensional, over an arbitrary field k unless otherwise stated.

Definition chasing questions

1. Let $\mathfrak{g}$ be the real vector space $\mathbb{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) and verify that this makes $\mathfrak{g}$ into a Lie algebra. Write down the multiplication table for $\mathfrak{g}$ relative to the usual basis for $\mathbb{R}^3$.

2. Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis for $\mathfrak{sl}_2(k)$. Write down the multiplication table for $\mathfrak{sl}_2(k)$ relative to this basis. Hence, compute the matrices of $\operatorname{ad} e, \operatorname{ad} h$ and $\operatorname{ad} f$ with respect to this basis.

3. (a) Show that there is precisely one Lie algebra of dimension 1 (up to isomorphism).

(b) Show that there are precisely two non-isomorphic Lie algebras of dimension 2 (one is abelian, the other is not).

(c) Let $\mathfrak{g}$ be the Lie algebra over k with basis $\{x, y, z\}$ and relations $[xy] = z, [yz] = x, [zx] = y$ (compare with question 1). If $k = \mathbb{C}$, show that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(k)$, but that this is *false* if $k = \mathbb{R}$.

(So, the classification of 3-dimensional Lie algebras depends on the ground field k.)

4. Prove that the centre of $\mathfrak{gl}_n(k)$ is the set of scalar matrices. Prove that $\mathfrak{sl}_n(k)$ has centre 0, unless $char k$ divides n, in which case the centre is again the set of scalar matrices.

5. Show $\mathfrak{sl}_2(k)$ is simple if $char k \neq 2$. What happens if $char k = 2$?

(*Hints:* Work in the basis of question 2. By applying $\operatorname{ad} e$ twice or $\operatorname{ad} f$ twice, show that if $0 \neq ae + bf + ch$ lies in an ideal I of $\mathfrak{sl}_2(k)$ then one of e, f or h lies in I, hence $I = \mathfrak{sl}_2(k)$.)

Derivations

6. Prove that the set of all inner derivations $\operatorname{ad} x, x \in \mathfrak{g}$ is an ideal of $\operatorname{Der} \mathfrak{g}$.

7. Verify that the commutator of two derivations of a k-algebra is again a derivation. Is the ordinary product always a derivation?

The PBW theorem

8. If $\mathfrak{g}$ is a free Lie algebra on a set X, show that $U(\mathfrak{g})$ is isomorphic to $T(V)$, where V is the vector space with X as basis.

9. Describe the free Lie algebra on the set $X = \{x\}$.

10. Let $\mathfrak{g}$ be an arbitrary finite dimensional Lie algebra. Use the PBW theorem to show that $U(\mathfrak{g})$ has no zero divisors.

Soluble and nilpotent Lie algebras

11. Let $\mathfrak{u}_n(k)$ and $\mathfrak{b}_n(k)$ be the set of all strictly upper triangular (ie zeros on the diagonal) and upper triangular (ie anything on the diagonal) $n \times n$ matrices over k respectively. Show that these are Lie subalgebras of $\mathfrak{gl}_n(k)$.

12. Let $\mathfrak{g} = \mathfrak{u}_n(k)$ as in question 11. Show that the lower central series of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}^0 > \mathfrak{g}^1 > \mathfrak{g}^2 > \cdots > \mathfrak{g}^r = 0$$

where $\mathfrak{g}^s$ equals $\{M \in \mathfrak{gl}_n(k) | M_{i,j} = 0 \text{ for all } 1 \leq i, j \leq n \text{ with } j - s \leq i \leq n\}$. Deduce that $\mathfrak{u}_n(k)$ is nilpotent.

13. Using question 12, show that $\mathfrak{b}_n(k)$ is soluble.

14. Show $\mathfrak{sl}_2(k)$ is nilpotent if $char k = 2$.

15. Let $\mathfrak{g}$ be nilpotent and $\mathfrak{k}$ be a proper subalgebra of $\mathfrak{g}$. Show that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{k})$ is strictly larger than $\mathfrak{k}$.

16. Let k be a field of characteristic $p > 0$. Let $x, y \in \mathfrak{gl}_p(k)$ be the following $p \times p$ matrices:

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, y = \operatorname{diag}(0, 1, \ldots, p-1).$$

Show that x, y generate a 2-dimensional soluble subalgebra of $\mathfrak{gl}_p(k)$ but that they have no common eigenvector. Hence, Lie's theorem is false in general in non-zero characteristic.

The Killing form

17. Using question 2, compute the Killing form explicitly for $\mathfrak{sl}_2(\mathbb{C})$ and hence verify directly that it is non-degenerate on $\mathfrak{sl}_2(\mathbb{C})$.

18. Let k have characteristic 3. Show $\mathfrak{sl}_3(k)$ modulo its centre is semisimple but has degenerate Killing form.

19. Let $char k = p \neq 0$. Prove that $\mathfrak{g}$ is semisimple if its Killing form is non-degenerate (the converse fails by example 18).

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