

## CHAPTER 2

# Chevalley groups

### 1. The main construction

Now we'll assume  $V_k = k \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$  is the reduction modulo  $p$  of some faithful finite dimensional  $\mathfrak{g}$ -module via some choice of admissible lattice. We wish to study automorphisms of  $V_k$  of the form

$$x_{\alpha}(t) := \exp tx_{\alpha} = \sum_{n=0}^{\infty} t^n \frac{x_{\alpha}^n}{n!}$$

for  $t \in k$  and  $\alpha \in \Phi$ . Of course the right hand side doesn't make sense until we explain how to interpret it. One way is to replace  $\frac{x_{\alpha}^n}{n!}$  with the element  $x_{\alpha}^{(n)} = 1 \otimes \frac{x_{\alpha}^n}{n!}$  of  $U_k$ , which does make sense as an endomorphism of  $U_k$ . Then  $x_{\alpha}(t)$  makes perfect sense, because all but finitely many  $x_{\alpha}^{(n)}$  act as zero (since  $V_k$  is the direct sum of its weight spaces). Another way is to consider first  $\exp Tx_{\alpha}$  as a map from  $V_{\mathbb{Z}}$  to  $\mathbb{Z}[T] \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$ ,  $T$  an indeterminate. That makes sense. Hence  $1 \otimes \exp Tx_{\alpha}$  is a map from  $V_k$  to  $k \otimes_{\mathbb{Z}} \mathbb{Z}[T] \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$ . Finally composing this map with the evaluation of  $T$  at  $t \in k$ , we get a map  $\exp tx_{\alpha}$  from  $V_k$  to  $V_k$ .

Let  $X_{\alpha}$  be the *root group*  $\{x_{\alpha}(t) \mid t \in k\}$ . Clearly  $x_{\alpha}(t)x_{\alpha}(s) = x_{\alpha}(t+s)$ , so  $X_{\alpha}$  is certainly a quotient of the additive group of the field (we'll see soon that actually  $X_{\alpha} \cong (k, +)$ ). Let  $G$  be the subgroup of  $GL(V_k)$  generated by all  $X_{\alpha}$ 's for  $\alpha \in \Phi$ . We call it the *Chevalley group* (associated to  $\mathfrak{g}, V, k$  and the choice  $V_{\mathbb{Z}}$  of admissible lattice). We will show in a while that  $G$  actually only depends on  $\mathfrak{g}, k$  and the subgroup  $A$  of the weight lattice  $P$  generated by  $\Pi(V)$ ... Recall  $A$  for sure includes the root lattice  $Q$ . That is helpful to know when thinking about examples, but remember we didn't prove it yet.

Now I want to give some trivial but helpful examples, taking  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ . Consider first  $V$  to be the natural 2-dimensional module and  $V_{\mathbb{Z}} = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ . Let  $\alpha$  be the positive root as usual. Then

$$x_{\alpha}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, x_{-\alpha}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

and it's an easy exercise which I'll do in class to see that these generate the group  $SL_2(k)$ . This is the case when  $A = P$ .

Instead, take  $V$  to be the adjoint representation  $\mathfrak{g}$  itself and  $V_{\mathbb{Z}} = \mathbb{Z}e \oplus \mathbb{Z}h \oplus \mathbb{Z}f$ . In this basis, we have that

$$x_{\alpha}(t) = \begin{pmatrix} 1 & -2t & -t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, x_{-\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -t^2 & 2t & 1 \end{pmatrix}.$$

Remembering that  $V_k = \mathfrak{sl}_2(k)$ , these are the same matrices as the linear maps  $\mathfrak{sl}_2(k) \rightarrow \mathfrak{sl}_2(k)$  defined by conjugating by the  $2 \times 2$  matrices above. This is the group  $PSL_2(k)$ , which by definition is the image of  $SL_2(k)$  under its representation by conjugation on  $2 \times 2$  matrices (observe the scalar matrices act as 1...). This is the case  $A = Q$ . In fact, these are the only two Chevalley groups you can get by starting with  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ , because in this case  $P/Q = \mathbb{Z}_2$ .

In general, when  $A = P$ , we call  $G$  the *simply connected* or *universal* Chevalley group of type  $\mathfrak{g}$  over the field  $k$ . When  $A = Q$  (which is always the case e.g. if  $V$  is the adjoint representation) then  $G$  is the *adjoint* Chevalley group over  $k$ .

Here is an approximate table giving the possible Chevalley groups you can get by varying  $\mathfrak{g}$  and  $A$ .

$\mathfrak{g}$	$P/Q$	$A = P$	intermediate $A$	$A = Q$
$A_1$	$\mathbb{Z}_2$	$SL_2(k)$		$PSL_2(k)$
$A_{n-1}$	$\mathbb{Z}_n$	$SL_n(k)$	$SL_n(k)/\text{central}$	$PSL_n(k)$
$C_n$	$\mathbb{Z}_2$	$Sp_{2n}(k)$		$PSp_{2n}(k)$
$B_n$	$\mathbb{Z}_2$	$Spin_{2n+1}(k)$		$SO_{2n+1}(k)$
$D_n$ $n$ even	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$Spin_{2n}(k)$	$SO_{2n}(k)$	$PSO_{2n}(k)$
$D_n$ $n$ odd	$\mathbb{Z}_4$	$Spin_{2n}(k)$	$SO_{2n}(k)$	$PSO_{2n}(k)$
$E_6$	$\mathbb{Z}_3$	$\hat{E}_6(k)$		$E_6(k)$
$E_7$	$\mathbb{Z}_2$	$\hat{E}_7(k)$		$E_7(k)$
$G_2, F_4, E_8$	1		$G_2(k), F_4(k), E_8(k)$	

Here are some comments about this approximate table.

First, when  $A = P$  the group  $G$  is the simply connected one, when  $A = Q$  the group  $G$  is the adjoint one. Often you can distinguish these by looking at the center  $Z(G)$  – often (but not always if the characteristic  $p$  of  $k$  is too small) this is isomorphic to  $A/Q$ . For instance for  $SL_n(k)$  when  $p \nmid n$ , the center of  $SL_n(k)$  is the group  $\mu_n$  of  $n$ th roots of unity, which is cyclic of order  $n$ . For each subgroup of that, you get a central normal subgroup, and can take the quotient to get one of the intermediate Chevalley groups between  $SL_n(k)$  and  $PSL_n(k)$ . If you factor out the whole center you get  $PSL_n(k)$ .

Second, when  $k$  is algebraically closed, the table is not approximate, it is correct. The groups  $SO_n(k)$  and  $Sp_{2n}(k)$  in these cases you should know (the isometries of a non-degenerate symmetric resp. skew symmetric bilinear form). Probably you've never seen the right definition of  $SO_n(k)$  when  $k$  has characteristic 2 (you need both a bilinear form and a quadratic form to get it right). The groups  $PSO_n(k)$  and  $PSp_{2n}(k)$  are these modulo their centers (usually  $\pm 1$ ). Note when  $n$  is odd that the matrix  $-1$  is not of determinant 1, which is why  $SO_{2n+1}(k)$  has no center... there is no  $PSO_{2n+1}(k)$  in the table in this case! The Spin groups (the simply connected Chevalley groups) are a little harder to define – of course we get them from our construction by taking  $V$  to be a spin representation (the last fundamental dominant weight). Its good enough to know for our purposes that  $SO_n(k)$  is (usually) a double cover of the  $SO_n(k)$ .

Now for the reason the table is approximate. Unfortunately it is wrong in the  $B_n$  and  $D_n$  rows when  $k$  is not algebraically closed. In that case, you need to replace the groups I've written with their *commutator subgroups* (which might in

these cases be a little bit smaller depending on the field). For example where I've written  $SO_{2n+1}(k)$  I should really have written  $\Omega_{2n+1}(k)$ , the commutator subgroup of the special orthogonal group of isometries of a  $(2n+1)$ -dimensional vector space equipped with a non-degenerate quadratic form of Witt index  $n$ . Where I've written  $SO_{2n}(k)$  I should really have written  $\Omega_{2n}^+(k)$ , the commutator subgroup of the special orthogonal group of isometries of a  $(2n)$ -dimensional vector space equipped with a non-degenerate quadratic form of Witt index  $n$ .

These are things that we are not going to study here and which are not important for what we're going to do, so I don't feel too uncomfortable about being a bit crappy in this explanation. Its actually rather a technical place. My advice: ask James.

In class I will do a couple more examples writing down explicitly the matrices that give some of the root groups for  $B_2$  and  $C_2$ . The point is: the Chevalley construction gives some completely concrete matrix groups. Its just that apart from some trivial cases the matrices are too big for this to be terribly helpful.

## 2. Chevalley's commutator formula

We'll need now to use the following lemma that you've seen before more than once:

LEMMA 2.1. *Suppose that  $A$  and  $B$  are elements of an associative algebra, let  $l_A$  be left multiplication by  $A$ ,  $r_A$  be right multiplication by  $A$  and  $\text{ad } A$  be the map  $l_A - r_A$  (all are endomorphisms of the given associative algebra). Suppose  $\exp \text{ad } A$ ,  $\exp l_A$ ,  $\exp r_A$  and  $\exp A$  all make sense. Then,*

$$(\exp A)B(\exp(-A)) = (\exp \text{ad } A)(B).$$

PROOF. We've seen this before: since  $l_A$  and  $r_A$  commute,

$$\exp \text{ad } A = \exp l_A \exp(-r_A) = l_{\exp A} r_{\exp(-A)}.$$

□

Now we come to the first important theorem about Chevalley groups.

THEOREM 2.2. *Let  $\alpha, \beta$  be roots with  $\alpha + \beta \neq 0$  Then in the ring of formal power series  $U_{\mathbb{Z}}[[t, u]]$ , we have the identity*

$$(\exp t x_{\alpha}, \exp u x_{\beta}) = \prod \exp c_{i,j} t^i u^j x_{i\alpha + j\beta}$$

where  $(A, B) = ABA^{-1}B^{-1}$ , the product on the right is over all roots  $i\alpha + j\beta$  for  $i, j > 0$  arranged in some fixed order, and the  $c_{i,j}$ 's are integers depending on  $\alpha, \beta$  and the chosen ordering but not on  $t$  or  $u$ . Moreover,  $c_{1,1}$  is the same as the structure constant  $[x_{\alpha}, x_{\beta}] = \pm(r+1)x_{\alpha+\beta}$  from our original choice of Chevalley basis.

PROOF. (Sketch) In  $U(\mathfrak{g})[[t, u]]$ , set

$$f(t, u) = (\exp t x_{\alpha}, \exp u x_{\beta}) \prod \exp(-c_{i,j} t^i u^j x_{i\alpha + j\beta}).$$

The product on the right is taken over  $i, j > 0$  such that  $i\alpha + j\beta \in \Phi$  and in the reverse of the fixed order. The coefficients  $c_{i,j}$  are some complex numbers which

are still to be determined. In fact I'll treat them like indeterminates to start with then we'll see how to define them so that  $f(t, u) = 1$ , which is of course what we need to do in order to prove the theorem.

Note that  $t \frac{d}{dt} \exp tx_\alpha = tx_\alpha \exp tx_\alpha$ . So by the product rule we get that

$$\begin{aligned} t \frac{d}{dt} f(t, u) &= tx_\alpha f(t, u) + \\ &(\exp tx_\alpha)(\exp ux_\beta)(-tx_\alpha)(\exp(-tx_\alpha))(\exp(-ux_\beta)) \prod \exp(-c_{i,j} t^i u^j x_{i\alpha+j\beta}) \\ &+ \sum_{k,l} (\exp tx_\alpha, \exp ux_\beta) \prod_{i\alpha+j\beta > k\alpha+l\beta} \exp(-c_{i,j} t^i u^j x_{i\alpha+j\beta}) \times \\ &(-c_{k,l} k t^k u^l x_{k\alpha+l\beta}) \times \prod_{i\alpha+j\beta \leq k\alpha+l\beta} \exp(-c_{i,j} t^i u^j x_{i\alpha+j\beta}). \end{aligned}$$

Now we bring the terms  $-tx_\alpha$  and  $-c_{k,l} k t^k u^l x_{k\alpha+l\beta}$  (\*) to the front using relations like

$$(\exp ux_\beta)(-tx_\alpha) = (\exp \text{ad } ux_\beta)(-tx_\alpha)(\exp ux_\beta)$$

and

$$(\exp \text{ad } ux_\beta)(-tx_\alpha) = -tx_\alpha - n_{\beta,\alpha} t u x_{\alpha+\beta} - \dots$$

where  $[x_\beta, x_\alpha] = n_{\beta,\alpha} x_{\alpha+\beta}$ . The result is an expression of the form  $Af(t, u)$  with  $A \in \mathfrak{g}[[t, u]]$ .

Now, think how you can get  $c_{k,l}$  as a coefficient (not inside an exp) when you do this. You get it from the term (\*) but otherwise it only comes when terms to the right of (\*) are pulled passed  $\exp(-c_{k,l} k t^k u^l x_{k\alpha+l\beta})$ . These produce coefficients involving  $c_{k,l}$  but of degree  $> (k+l)$  in  $t$  and  $u$ . Thus,

$$A = \sum_{k,l \geq 1} (-k c_{k,l} + p_{k,l}) t^k u^l x_{k\alpha+l\beta}$$

where  $p_{k,l}$  is a polynomial in the  $c_{i,j}$ 's for which  $i+j < k+l$ .

At last we can now inductively determine the values of  $c_{k,l} \in \mathbb{C}$  using the lexicographic ordering on pairs  $(i, j)$ , to ensure that  $A = 0$ . Then  $t \frac{d}{dt} f(t, u) = 0$  implies that  $f(t, u) = f(0, u) = 1$  and we're almost done.

It remains to show that the  $c_{i,j}$ 's are integers (and to compute  $c_{1,1}$ ). The idea for that is to compute the coefficient of  $t^i u^j$  in the commutator formula obtained so far (which for sure belongs to  $U_{\mathbb{Z}}[[t, u]]$ ). It looks like  $-c_{i,j} x_{i\alpha+j\beta}$  plus terms coming from exponentials of multiples of  $x_{k\alpha+l\beta}$  with  $k+l < i+j$ . Now you use induction to see that  $c_{i,j} x_{i\alpha+j\beta}$  lies in  $U_{\mathbb{Z}}$ , hence  $c_{i,j} \in \mathbb{Z}$  by the basis theorem for  $U_{\mathbb{Z}}$ .  $\square$

**COROLLARY 2.3.** *Replacing  $\exp tx_\alpha$  etc... in the theorem with  $x_\alpha(t)$ , then the resulting equation holds in the Chevalley group  $G$ .*

For example, from this formula in  $SL_3(k)$ , taking  $\alpha = \varepsilon_1 - \varepsilon_2$  and  $\beta = \varepsilon_2 - \varepsilon_3$ , you get that

$$(x_\alpha(t), x_\beta(u)) = x_{\alpha+\beta}(tu),$$

which you can easily check directly. Of course its more complicated for  $B_2$  and  $G_2$ ...

We call a set  $R$  of roots *closed* if  $\alpha, \beta \in R$ ,  $\alpha + \beta \in \Phi$  implies  $\alpha + \beta \in R$ . For example:  $R = \Phi^+$ ,  $R = \Phi^+ \setminus \{\alpha_i\}$  and  $R = \{\alpha \in \Phi^+ \mid \text{ht}(\alpha) \geq r\}$  are all closed sets of roots.

We call a subset  $I$  of a closed set  $R$  an *ideal* if  $\alpha \in I, \beta \in R$  implies  $\alpha + \beta \in I$ . The above examples are all ideals in  $\Phi^+$ .

LEMMA 2.4. *Let  $I$  be an ideal in a closed set  $R$ . Assume  $R \cap (-R) = \emptyset$ . Let  $X_R$  and  $X_I$  be the groups generated by all  $X_\alpha$  for  $\alpha \in R$  and  $\alpha \in I$ , respectively. Then  $X_I \trianglelefteq X_R$ .*

PROOF. Immediate from commutator formula. □

LEMMA 2.5. *Let  $R$  be a closed set of roots such that  $R \cap (-R) = \emptyset$ . Then every element of  $X_R$  can be written uniquely as  $\prod_{\alpha \in R} x_\alpha(t_\alpha)$  for  $t_\alpha \in k$ , where the product is taken in any fixed order.*

PROOF. I'll prove this just under the assumption the fixed order is compatible with height, i.e.  $\text{ht}(\alpha) < \text{ht}(\beta)$  implies  $\alpha < \beta$ . The general case can be reduced to this case with a little more work...

Let  $\alpha$  be the first element of  $R$ . Let  $I = R \setminus \{\alpha\}$ . Its an ideal in  $R$ . So,  $X_R = X_\alpha X_I$ . By induction on  $|R|$ , every element of  $X_I$  is the product of the  $x_\beta(t_\beta)$  for  $\beta \in I$  with uniqueness of expression.

Now suppose  $y \in X_R$  written as  $y = x_\alpha(t) \prod x_\beta(t_\beta)$ . Since  $V$  was faithful, we can find a weight vector  $v \in V_k$  of weight  $\lambda$  such that  $x_\alpha v \neq 0$ . Then,  $yv = v + tx_\alpha v + z$  where  $v$  is of weight  $\lambda$ ,  $x_\alpha v$  is of weight  $\lambda + \alpha$  and  $z$  is a sum of terms from other weight spaces. Hence  $t$  is uniquely determined by  $y$ . The remaining  $t_\beta$ 's are uniquely determined by induction applied to  $x_\alpha(-t)y = \prod x_\beta(t_\beta) \in X_I$ . □

This lemma has lots of consequences:

- (1) The map  $t \mapsto x_\alpha(t)$  is an isomorphism from  $(k, +)$  onto  $X_\alpha$ . (Proof: Apply lemma with  $R = \{\alpha\}$ .)
  - (2) Let  $U = X_{\Phi^+}$ . Then,  $U = \prod_{\alpha \in \Phi^+} X_\alpha$  with uniqueness of expression, where the product is taken in any fixed order.
  - (3) The subgroup  $U$  of  $G$  consists of upper uni-triangular matrices relative to an appropriately chosen basis of  $V_k$ . In particular, every entry of  $U$  is a unipotent matrix. (Proof: Pick a basis of weight vectors ordered in some way refining the dominance ordering.)
  - (4) For  $i \geq 1$ , let  $U_i$  be the group generated by all  $X_\alpha$  for  $\text{ht}(\alpha) \geq i$ . Then,  $U_i$  is normal in  $U$  and  $(U, U_i) \subseteq U_{i+1}$ , hence  $U$  is a nilpotent group.
  - (5) If  $\Phi^+ = R \sqcup S$  with  $R$  and  $S$  closed, then  $U = X_R X_S$  and  $X_R \cap X_S = \{1\}$ .
- You should think what these things are saying when  $G = SL_n(k)$ ...

### 3. The basic relations

So far we've defined elements  $x_\alpha(t)$  and proved Chevalley's commutator formula explaining how to commute  $x_\alpha(t)$  with  $x_\beta(u)$  when  $\alpha + \beta \neq 0$ . These were all unipotent matrices, i.e. all eigenvalues were 1. Now we introduce some semisimple elements of  $G$  (i.e. diagonalizable matrices) too. For  $\alpha \in \Phi$  and  $t \in k^\times$  define

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t),$$

and

$$h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}.$$

Sometimes I'll write simply  $w_\alpha$  for  $w_\alpha(1)$ . For example, if  $G = SL_2(k)$  and  $\alpha$  is the positive root, then

$$w_\alpha(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \quad h_\alpha(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

I'll refer to these as the ‘‘monomial elements’’ and ‘‘diagonal elements’’ respectively.

LEMMA 3.1. *For any roots  $\alpha, \beta \in \Phi$  and  $t \in k^\times$ :*

- (a)  $w_\alpha(t)x_\beta w_\alpha(t)^{-1} = ct^{-\langle \beta, \alpha \rangle} x_{s_\alpha(\beta)}$  where  $c = c(\alpha, \beta) = \pm 1$  independent of  $t, k$  and  $V$ , and  $c(\alpha, \beta) = c(\alpha, -\beta)$ . Here  $s_\alpha$  is the reflection in the hyperplane  $H_\alpha$ .
- (b) For  $v \in V_k$  of weight  $\mu$ , there exists  $v' \in V_k$  of weight  $s_\alpha(\mu)$  independent of  $t$  such that  $w_\alpha(t)v = t^{-\langle \mu, \alpha \rangle} v'$ .
- (c)  $h_\alpha(t)$  acts on the  $\mu$ -weight space of  $V_k$  by multiplication by  $t^{\langle \mu, \alpha \rangle}$ .

PROOF. I'll prove this assuming the ground field is  $\mathbb{C}$ . In fact everything then follows at the level of power series in  $U_{\mathbb{Z}}[[t]]$  or as endomorphisms of  $V_{\mathbb{Z}}[[t]]$ , so you get it automatically over an arbitrary field too.

We first show that  $w_\alpha(t)hw_\alpha(t)^{-1} = s_\alpha(h)$  for each  $h \in \mathfrak{h}$ . (Note I'm viewing the left hand side just as some matrices, i.e. endomorphisms of  $V$ , and the right hand side is the natural action of  $s_\alpha \in W$  on  $\mathfrak{h}$ .) By linearity, it suffices to check this just when  $h = h_\alpha$  (because  $\mathfrak{h} = \mathbb{C}h_\alpha \oplus \ker \alpha$  and  $w_\alpha(t)$  commutes with all elements of  $\ker \alpha$ ). But then  $x_\alpha(t)$  hence  $w_\alpha(t)$  only involves elements of the three dimensional algebra  $\langle x_\alpha, y_\alpha, h_\alpha \rangle$ , so we can check the thing we're after just by calculations in this algebra. Take the usual representation of  $\mathfrak{sl}_2$ ... then we know exactly what  $2 \times 2$  matrices everything is and its easy.

Now we prove (b). From the definitions of  $x_\alpha(t)$  and  $w_\alpha(t)$ , it follows that if

$$v'' = w_\alpha(t)v$$

then

$$v'' = \sum_{i \in \mathbb{Z}} t^i v_i$$

for  $v_i$  of weight  $\mu + i\alpha$ . But acting with  $h \in \mathfrak{h}$  and using the previous paragraph,  $hv'' = (s_\alpha(\mu))(h)v''$ , so  $v''$  lies in the  $s_\alpha(\mu)$ -weight space. Hence the only non-zero  $v_i$  is when  $i = -\langle \mu, \alpha \rangle$ .

Now apply (b) to the adjoint representation (using Lemma 2.1) to get that  $w_\alpha(t)x_\beta w_\alpha(t)^{-1} = ct^{-\langle \beta, \alpha \rangle} x_{s_\alpha(\beta)}$  where  $c \in \mathbb{C}$  is independent of  $t$  and  $V$ . But taking  $t = 1$ , the left hand side lies in  $U_{\mathbb{Z}}$  so actually  $c \in \mathbb{Z}$ , actually  $c = \pm 1$  since  $w_\alpha(1)$  is invertible. Finally,  $h_{s_\alpha(\beta)} = w_\alpha h_\beta w_\alpha^{-1} = [w_\alpha x_\beta w_\alpha^{-1}, w_\alpha x_{-\beta} w_\alpha^{-1}] = c(\alpha, \beta)c(\alpha, -\beta)h_{s_\alpha(\beta)}$  so  $c(\alpha, \beta) = c(\alpha, -\beta)$ . This proves (a).

To get (c), note that  $w_\alpha(t)^{-1} = w_\alpha(-t)$ , so  $h_\alpha(t) = w_\alpha(-t)^{-1}w_\alpha(-1)$ . By (b),  $w_\alpha(-t)v = (-t)^{-\langle \mu, \alpha \rangle} v'$  and  $w_\alpha(-1)v = (-1)^{-\langle \mu, \alpha \rangle} v'$ . Hence  $w_\alpha(-t)^{-1}w_\alpha(-1)v = t^{\langle \mu, \alpha \rangle} v$ .  $\square$

COROLLARY 3.2. *For  $\alpha, \beta \in \Phi$  and  $t \in k^\times$ :*

- (a)  $w_\alpha h_\beta(t) w_\alpha^{-1} = h_{s_\alpha(\beta)}(t)$ .
- (b)  $w_\alpha x_\beta(t) w_\alpha^{-1} = x_{s_\alpha(\beta)}(ct)$  with  $c$  as in Lemma 3.1(a).
- (c)  $h_\alpha(t) x_\beta(u) h_\alpha(t)^{-1} = x_\beta(t^{(\beta, \alpha)} u)$ .

PROOF. For (a), apply both sides to a vector  $v \in V_k$  of weight  $\mu$  and use the lemma.

For (b), we know that  $w_\alpha x_\beta w_\alpha^{-1} = c x_\beta$ , now exponentiate.

For (c), we know by (c) of the lemma applied to the adjoint representation that  $h_\alpha(t) x_\beta h_\alpha(t)^{-1} = t^{(\beta, \alpha)} x_\beta$ . Now exponentiate.  $\square$

We have now established all of the following relations.

- (R1)  $x_\alpha(t) x_\alpha(u) = x_\alpha(t + u)$ .
- (R2)  $(x_\alpha(t), x_\beta(u)) = \prod x_{i\alpha + j\beta}(c_{i,j} t^i u^j)$  (if  $\alpha + \beta \neq 0$ ), where the  $c_{i,j}$ 's are as in Chevalley's commutator formula.
- (R3)  $w_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t)$ .
- (R4)  $h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1}$ .
- (R5)  $w_\alpha = w_\alpha(1)$ .
- (R6)  $w_\alpha h_\beta(t) w_\alpha^{-1} = h_{s_\alpha(\beta)}(t)$ .
- (R7)  $w_\alpha x_\beta(t) w_\alpha^{-1} = x_{s_\alpha(\beta)}(ct)$  with  $c$  as in Lemma 3.1.
- (R8)  $h_\alpha(t) x_\beta(u) h_\alpha(t)^{-1} = x_\beta(t^{(\beta, \alpha)} u)$ .

Observe all the relations are independent of  $V$ . So all results proved using (R1)–(R8) will also be independent of  $V$ ...

Let  $H$  be the subgroup of  $G$  generated by all  $h_\alpha(t)$ 's. This is an abelian group since all  $h_\alpha(t)$ 's act diagonally on  $V_k$ . Indeed, it is a subgroup of the group of diagonal matrices with respect to any weight basis of  $V_k$ . Let  $N$  be the subgroup of  $G$  generated by all  $w_\alpha(t)$ 's. Recall also that  $U$  is the subgroup generated by all  $x_\alpha(t)$ 's for  $\alpha \in \Phi^+$ . Let  $B$  be the group generated by  $H$  and  $U$ . Keep in mind that if  $G = SL_n(k)$  then  $H$  is the diagonal matrices of determinant 1,  $U$  is upper unitriangular matrices,  $B$  is all upper triangular matrices of determinant 1, and  $N$  is the monomial matrices (matrices of determinant 1 with exactly one non-zero entry in every row and column).

LEMMA 3.3.  *$U$  is normal in  $B$ ,  $B = UH$  and  $U \cap H = \{1\}$ . Hence  $B$  is the semidirect product of  $U$  by  $H$ .*

PROOF. The fact that  $U$  is normal in  $B$  follows by (R8). Relative to a suitable basis for  $V_k$ , any element of  $U \cap H$  is both diagonal and unipotent, hence its 1.  $\square$

LEMMA 3.4. *For  $\alpha \neq \beta$ ,  $X_\alpha \neq X_\beta$ .*

PROOF. If both  $\alpha, \beta$  are positive roots, this follows from Lemma 2.5. If one is positive and one is negative, it follows because we know  $V$  has a basis with respect to which  $X_\alpha$  is upper unitriangular and  $X_\beta$  is lower unitriangular.  $\square$

For the next lemma, we need to use a little information about Weyl groups:  $W$  is generated by the elements  $s_\alpha$  ( $\alpha \in \Phi$ ) subject only to the relations  $s_\alpha^2 = 1$  and  $s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)}$  (for all  $\alpha, \beta \in \Phi$ ). I'm not going to prove that here (it has nothing to do with Chevalley groups).

LEMMA 3.5.  *$H$  is normal in  $N$ , and there exists an isomorphism  $\phi : W \rightarrow N/H$  such that  $\phi(s_\alpha) = Hw_\alpha(t)$  for all roots  $\alpha$ .*

PROOF. By (R6), conjugation by  $w_\alpha$  preserves  $H$  and by (R4) and (R5),  $w_\alpha(t) = h_\alpha(t)w_\alpha$ . Hence  $H$  is normal in  $N$ .

Since  $Hw_\alpha(t) = Hw_\alpha$ , the coset  $Hw_\alpha(t)$  is independent of  $t$ . Since  $w_\alpha(1)w_\alpha(-1)$ , this implies that  $(Hw_\alpha(t))^2 = 1$ . Also  $w_\alpha w_\beta w_\alpha^{-1} = w_\alpha x_\beta(1)x_{-\beta}(-1)x_\beta(1)w_\alpha^{-1}$  which equals by (R7)  $x_{s_\alpha(\beta)}(c)x_{-s_\alpha(\beta)}(-c)x_{s_\alpha(\beta)}(c) = w_{s_\alpha(\beta)}(c)$ . So

$$(Hw_\alpha)(Hw_\beta)(Hw_\alpha) = Hw_{s_\alpha(\beta)}.$$

This verifies the relations of  $W$  hold, so there exists a surjective homomorphism  $\phi : W \rightarrow N/H$ . Finally, suppose  $w \in W$  lies in  $\ker \phi$ . Write  $w = s_{\alpha_1} \cdots s_{\alpha_n}$ , a product of reflections. Then applying  $\phi$  we get that  $w_{\alpha_1}(1) \cdots w_{\alpha_n}(1) \in H$ . Hence it normalizes each  $X_\alpha$ , but also it conjugates  $X_\alpha$  to  $X_{w(\alpha)}$ . Hence  $X_\alpha = X_{w(\alpha)}$  for all roots  $\alpha$ . By Lemma 3.4 this implies that  $w(\alpha) = \alpha$  for all  $\alpha$ . This implies  $w = 1$  (because it acts faithfully on its reflection representation).  $\square$

#### 4. The Bruhat decomposition

Recall the groups  $X_\alpha, G, B, U, H, N$  and  $W$ . Also we have an isomorphism  $W \cong N/H$ . I'll identify  $W$  with  $N/H$  via this isomorphism. From now on for each  $w \in W$ , I'll fix a choice of representative  $\dot{w} \in N$  for the coset  $w \in N/H$ . We may do this so that  $\dot{s}_\alpha = w_\alpha$  for each  $\alpha \in \Phi$ . The goal now is to prove:

THEOREM 4.1 (Bruhat, Chevalley).  *$G$  is the disjoint union of the double cosets  $B\dot{w}B$  for  $w \in W$ :*

$$G = \bigcup_{w \in W} B\dot{w}B.$$

Before I prove that, let us discuss the example  $G = SL_n(k)$  (a special case of the above theorem). Take any  $g \in SL_n(k)$ . Left multiplication by a matrix in  $U$  lets you add multiples of later rows to earlier rows. Right multiplication by a matrix in  $U$  lets you add multiples of earlier columns to later columns. By repeatedly applying such operations you can transform  $g$  to an element  $n \in N$ , i.e. to  $h \cdot w$  for some  $h \in H$  and  $w \in W$ . This proves that  $G = \bigcup_{w \in W} B\dot{w}B$ . We've at least started to prove the Bruhat decomposition in this case: it amounts to *Gaussian elimination*. It is an amusing exercise to finish of this elementary proof.

So you see – like everything else in this subject – we are generalizing some classical fact about  $n \times n$  matrices to an arbitrary representation of an arbitrary semisimple Lie algebra...

Now let us start to prove the theorem. It will take a while.

LEMMA 4.2. *If  $\alpha$  is a simple root then  $B \cup Bw_\alpha B$  is a group.*

PROOF. Let  $S = B \cup Bw_\alpha B$ . Since  $B$  is a group and  $w_\alpha^{-1}$  lies in the same coset of  $H$  as  $w_\alpha$  (since its image in  $W$  is an involution),  $S$  is closed under inversion. Since  $S^2 \subseteq BB \cap BBw_\alpha B \cup Bw_\alpha BB \cup Bw_\alpha BBw_\alpha B \subseteq S \cap Bw_\alpha Bw_\alpha B$  we just need to show that  $w_\alpha Bw_\alpha \subseteq S$ .

We first show  $X_{-\alpha} \subseteq S$ . If  $1 \neq y \in X_{-\alpha}$  there exists  $t \in k^\times$  such that  $y = x_{-\alpha}(t) = x_\alpha(t^{-1})w_\alpha(-t^{-1})x_\alpha(t^{-1}) \in Bw_\alpha B$ .

Now,  $w_\alpha B w_\alpha = w_\alpha U H w_\alpha^{-1} = w_\alpha X_\alpha X_{\Phi^+ \setminus \{\alpha\}} w_\alpha^{-1} H = X_{-\alpha} X_{\Phi^+ \setminus \{\alpha\}} H$  since  $s_\alpha$  leaves the set  $\Phi^+ \setminus \{\alpha\}$  invariant. This is contained in  $SB = S$ . We're done.  $\square$

LEMMA 4.3. *If  $w \in W$  and  $\alpha$  is a simple root, then*

- (a) *if  $w(\alpha) \in \Phi^+$  (i.e. if  $\ell(ws_\alpha) = \ell(w) + 1$ ) then  $B\dot{w}BBw_\alpha B = B\dot{w}w_\alpha B$ ;*
- (b) *in any case  $B\dot{w}BBw_\alpha B \subseteq B\dot{w}w_\alpha B \cup B\dot{w}B$ .*

PROOF. (a) Note that  $\dot{w}X_\alpha \dot{w}^{-1} \subseteq B$  and  $w_\alpha$  normalizes  $X_{\Phi^+ \setminus \{\alpha\}}$ . So

$$B\dot{w}BBw_\alpha B = B\dot{w}X_\alpha X_{\Phi^+ \setminus \{\alpha\}} w_\alpha B = B\dot{w}X_\alpha \dot{w}^{-1} \dot{w} w_\alpha w_\alpha^{-1} X_{\Phi^+ \setminus \{\alpha\}} w_\alpha B = B\dot{w}w_\alpha B.$$

(b) If  $w(\alpha) \in \Phi^+$  then we're done by (a). If  $w(\alpha) \in \Phi^-$  set  $w' = ws_\alpha$ , so  $w'(\alpha) \in \Phi^+$ . By (a) and the previous lemma,

$$\begin{aligned} B\dot{w}BBw_\alpha B &= B\dot{w}'w_\alpha BBw_\alpha B = B\dot{w}'BBw_\alpha BBw_\alpha B \\ &\subseteq B\dot{w}'B(B \cup Bw_\alpha B) = B\dot{w}'B \cup B\dot{w}'w_\alpha B. \end{aligned}$$

$\square$

COROLLARY 4.4. *If  $w \in W$  is written as  $w = s_{i_1} \cdots s_{i_l}$  (reduced expression) then*

$$BwB = Bw_{i_1} BBw_{i_2} B \cdots Bw_{i_l} B$$

where  $w_i = w_{\alpha_i}$  for short.

LEMMA 4.5.  *$G$  is generated by  $\{X_\alpha, w_\alpha \mid \alpha \in \Delta\}$ .*

PROOF. This follows since the simple reflections generate  $W$ , every root is  $W$ -conjugate to a simple root and  $w_\alpha X_\beta w_\alpha^{-1} = X_{s_\alpha(\beta)}$ .  $\square$

Now we can prove the theorem. First we show  $G = \bigcup_{w \in W} B\dot{w}B$ . By the previous lemma, the right hand side contains a set of generators for  $G$ . By the lemma before that, the product of two double cosets is another, hence the right hand side is closed under multiplication. Clearly its closed under inverses. So its a group, it must be all of  $G$ .

Now suppose  $B\dot{w}B = B\dot{w}'B$  for  $w, w' \in W$ . We show by induction on  $\ell(w)$  that  $w = w'$ . For the base case, if  $\ell(w) = 0$  then  $w = 1$ , so it says that  $\dot{w}' \in B$ . Hence  $\dot{w}'B(\dot{w}')^{-1} = B$ . This implies that  $w'(\Phi^+) = \Phi^+$ , hence  $w'(\Delta) = \Delta$ . Hence  $w' = 1$ , since  $W$  acts simply transitively on bases for  $\Phi$ .

Now say  $\ell(w) > 0$ . Pick  $\alpha \in \Delta$  so  $ws_\alpha$  is shorter than  $w$ . Then  $\dot{w}w_\alpha \in B\dot{w}'BBw_\alpha B \subseteq B\dot{w}'B \cup B\dot{w}'w_\alpha B = B\dot{w}'B \cup B\dot{w}'w_\alpha B$ . Induction hypothesis now gives either that  $w w_\alpha = w$ , which is a contradiction, or that  $w w_\alpha = w' w_\alpha$  hence  $w = w'$ .

The following theorem gives a more precise version of the Bruhat decomposition, giving a canonical form for elements of  $G$ .

THEOREM 4.6. *For  $w \in W$ , let  $U_w = U_{\Phi^+ \cap w^{-1}(\Phi^-)}$ . So  $U_w$  is the product of all root groups corresponding to positive roots sent to negative roots by  $w$ . Then, any element  $g$  of  $G$  can be written as*

$$g = uh\dot{w}'$$

for unique  $u \in U, h \in H, w \in W$  and  $u' \in U_w$ .

PROOF. Take  $g \in G$ . By Bruhat decomposition,  $g = b\dot{w}b'$  for unique  $w \in W$  and some  $b, b' \in B$ . But  $B = UH = HU$  and  $\dot{w}$  normalizes  $H$ . Hence  $h = uh\dot{w}u'$  for  $u \in U, h \in H$  and  $u' \in U$ . Now  $U = U'_w U_w$  where  $U'_w = U_{\Phi^+ \cap w^{-1}(\Phi^+)}$ , the product of root groups corresponding to positive roots left positive by  $w$ . So  $u' = u_1 u_2$  for  $u_1 \in U'_w$  and  $u_2 \in U_w$ , and you can commute  $u_1$  past  $\dot{w}$  to get something still lying in  $U$ . This shows that  $g = uh\dot{w}u'$  for  $u \in U, h \in U$  and  $u' \in U_w$ .

Now for uniqueness, supposed that  $u_1 h_1 \dot{w} u'_1 = u_2 h_2 \dot{w} u'_2$ . Then  $h_2^{-1} u_2^{-1} u_1 h_1 = \dot{w} u'_2 (u'_1)^{-1} \dot{w}^{-1}$ . The left hand side is an upper triangular matrix (for a suitable weight basis) and the right hand side is strictly lower triangular. Hence both sides are 1. This implies  $u_1 = u_2, h_1 = h_2, u'_1 = u'_2$ .  $\square$

## 5. Homomorphisms between Chevalley groups

Now we complete our analysis of Chevalley groups, by getting the desired independence of  $G$  on  $V, V_{\mathbb{Z}}$ . Recall  $A$  is the subgroup of  $P$  generated by the weights of  $V$ , and  $Q \subseteq A \subseteq P$ .

LEMMA 5.1. *Let  $G'$  be a group generated by elements  $x'_\alpha(t)$  ( $\alpha \in \Phi, t \in k$ ) such that the relations (R1)–(R8) all hold. Define subgroups  $U'$  and  $H'$  as we did for  $G$ .*

- (i) *Every element of  $U'$  can be written in the form  $\prod_{\alpha \in \Phi^+} x'_\alpha(t_\alpha)$  for  $t_\alpha \in k$  (product taken in some fixed order).*
- (ii) *For each  $w \in W$  fix an expression  $w = s_\alpha s_\beta \cdots$  as a product of reflections, and set  $\dot{w}' = w'_\alpha w'_\beta \cdots$  (where  $w'_\alpha = w'_\alpha(1)$ ). Then, every element of  $G'$  can be written as  $u'h'\dot{w}'v'$  for  $u' \in U, h' \in H, w \in W$  and  $v' \in U'_w$ .*

PROOF. Same as the proof for  $G$  – this part only depended on the relations.  $\square$

LEMMA 5.2. *Let  $G, G'$  be as above and suppose that  $\phi : G' \rightarrow G$  is a group homomorphism such that  $\phi(x'_\alpha(t)) = x_\alpha(t)$  for all  $\alpha, t$ . Then uniqueness of expression holds in (i) and (ii) of the previous lemma. Moreover,  $\ker \phi \subseteq Z(G') \subseteq H'$ . In particular,  $Z(G) \subseteq H$ .*

PROOF. The uniqueness of expression in  $G'$  follows easily from the uniqueness already established for  $G$ . (This was something proved for  $G$  that depended on more than just the relations (R1)–(R8) – we used the explicit representation of elements of  $G$  as matrices which is missing for  $G'$ .) For the last sentence, let  $x' = u'h'\dot{w}'v'$  lie in  $\ker \phi$ . Then  $1 = \phi(u')\phi(h')\dot{w}'\phi(v')$ . From this we get that  $w = 1$  so  $\dot{w}' = 1$ ,  $\phi(u') = 1$  and  $\phi(v') = 1$  so  $u' = v' = 1$ . So  $x' = h' = \prod h'_\alpha(t_\alpha)$  for some  $t_\alpha \in k^\times$ .

Now,  $x'x'_\beta(u)(x')^{-1} = x'_\beta(\prod_\alpha t_\alpha^{(\beta, \alpha)} u)$  by (R8). Applying  $\phi$  we get that  $\prod_\alpha t_\alpha^{(\beta, \alpha)} = 1$ . Hence  $x'$  commutes with  $x'_\beta(u)$ , so its central. Moreover, we have shown that  $\ker \phi \subseteq H'$ .

It just remains to show that  $Z(G) \subseteq H$ . If  $x = uh\dot{w}v \in Z(G)$  and  $w \neq 1$ , then there exists  $\alpha \in \Phi^+$  such that  $w(\alpha) \in \Phi^-$ . Then  $xx_\alpha(1) = x_\alpha(1)x$  contradicts the uniqueness in the Bruhat decomposition. Hence  $w = 1$  so  $x = uh$ . Let  $w_0$  be the element sending  $\Phi^+$  to  $\Phi^-$ . Then  $x = \dot{w}_0 x \dot{w}_0^{-1}$  is both upper and lower triangular, hence diagonal. Hence  $x = h$  lies in  $H$ .  $\square$

COROLLARY 5.3. *The relations (R1)–(R8) together with all the relations in  $H$  between the  $h_\alpha(t)$ 's forms a set of defining relations for  $G$ .*

PROOF. If the relations in  $H$  are imposed on  $H'$  then the map  $\phi$  in the lemma becomes an isomorphism.  $\square$

COROLLARY 5.4. *If  $G'$  is constructed as  $G$  from  $\mathfrak{g}$  and  $k$ , but using a different  $V'$  in place of  $V$ , then there exists a homomorphism from  $G'$  onto  $G$  such that  $\phi(x'_\alpha(t)) = x_\alpha(t)$  if and only if there exists a homomorphism  $\theta : H' \rightarrow H$  such that  $\theta(h'_\alpha(t)) = h_\alpha(t)$  for all  $\alpha$  and  $t$ .*

PROOF. Clearly if  $\phi$  exists then  $\theta$  exists. Conversely, assume  $\theta$  exists. Then the relations in  $H'$  form a subset of the relations in  $H$ . So we get that  $\phi$  exists by the previous corollary.  $\square$

So we just need to understand the group  $H$  better. Recall that  $h_\alpha(t)$  acts on the  $\mu$ -weight space of  $V_k$  as the scalar  $t^{\langle \mu, \alpha \rangle}$ .

- LEMMA 5.5. (a) *For each  $\alpha$ ,  $h_\alpha(t)$  is multiplicative as a function of  $t$ .*  
 (b)  *$H$  is an abelian group generated by the  $h_i(t)$ 's, where  $h_i(t) = h_{\alpha_i}(t)$ .*  
 (c)  *$\prod_{i=1}^{\ell} h_i(t_i) = 1$  if and only if  $\prod_{i=1}^{\ell} t_i^{\langle \mu, \alpha_i \rangle} = 1$  for all  $\mu \in A$ .*  
 (d) *The center of  $G$  is*

$$\left\{ \prod_{i=1}^{\ell} h_i(t_i) \mid \prod_{i=1}^{\ell} t_i^{\langle \beta, \alpha_i \rangle} = 1 \text{ for all } \beta \in Q \right\}.$$

*In particular,  $Z(G)$  is finite.*

PROOF. Parts (a) and (c) are pretty clear (note (c) is true for all  $\mu \in A$  if and only if it is true for all  $\mu \in \Pi(V)$ ). For (b), take any  $\alpha \in \Phi$  and write  $h_\alpha = \sum_{i=1}^{\ell} n_i h_i$ . Then I claim  $h_\alpha(t) = \prod h_i(t^{n_i})$ . To see this it is enough to check both sides act on the  $\mu$ -weight space of  $V$  by the same scalar. The left hand side acts by  $t^{\langle \mu, \alpha \rangle} = t^{\mu(h_\alpha)} = t^{\sum n_i \mu(h_i)} = \prod t^{n_i \mu(h_i)}$ . Yep. Finally for (d),  $\prod h_i(t_i)$  commutes with  $x_\beta(u)$  if and only if  $\prod_{i=1}^{\ell} t_i^{\langle \beta, \alpha_i \rangle} = 1$  by (R8). I'll leave you to puzzle over why this is a finite group. What system of equations have you got to solve?  $\square$

COROLLARY 5.6. *If  $A = P$ , then every  $h \in H$  can be written uniquely as  $h = \prod_{i=1}^{\ell} h_i(t_i)$  for  $t_i \in k^\times$ . If  $A = Q$  then  $Z(G) = 1$ .*

Finally we can complete the main discussion about Chevalley groups.

THEOREM 5.7. *Let  $G$  and  $G'$  be Chevalley groups constructed from  $\mathfrak{g}$ ,  $k$ , and choices  $V$  and  $V'$  of faithful finite dimensional representations, with weight groups  $A$  and  $A'$  respectively. If  $A' \supseteq A$  then there exists a homomorphism  $\phi : G' \rightarrow G$  such that  $x'_\alpha(t) \mapsto x_\alpha(t)$  and  $\ker \phi \subseteq Z(G')$ . In particular, if  $A' = P$  then  $G'$  covers all other Chevalley groups defined from  $\mathfrak{g}$  and  $k$  (its the universal Chevalley group) and if  $A = Q$  then  $G$  is covered by all other Chevalley groups defined from  $\mathfrak{g}$  and  $k$  (its the adjoint Chevalley group).*

PROOF. Just note that if  $A' \supseteq A$  then  $\prod_{i=1}^{\ell} t_i^{\langle \mu, \alpha_i \rangle} = 1$  for all  $\mu \in A'$  implies  $\prod_{i=1}^{\ell} t_i^{\langle \mu, \alpha_i \rangle} = 1$  for all  $\mu \in A$ . In other words, by the previous lemma, all the relations in  $H$  are satisfied in  $H'$  too, so there is a homomorphism  $\theta : H' \rightarrow H$  such that  $h'_i(t) \mapsto h_i(t)$  for each  $i$ . This also maps  $h'_\alpha(t)$  to  $h_\alpha(t)$  for each  $\alpha$ . So there's a  $\phi$  by Corollary 5.4.  $\square$

COROLLARY 5.8. *For each  $\alpha \in \Phi$ , there is a group homomorphism*

$$\phi_\alpha : SL_2(k) \rightarrow G$$

*under which*

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-\alpha}(t)$$

*for each  $t \in k$ .*

PROOF. Apply the theorem to the three dimensional subalgebra of  $\mathfrak{g}$  spanned by  $x_\alpha, h_\alpha, y_\alpha$  but keeping the representation  $V$  as before. You get that the subgroup of  $S_\alpha$  of  $G$  is a quotient of the universal Chevalley group of type  $A_1$ , i.e.  $SL_2(k)$ .  $\square$

That concludes the first look at Chevalley groups. In the next chapter I'm going to talk about some fun things that should at least make sense after this introduction. Here are a couple of exercises you could try.

(1) If  $G$  is of universal type, then the homomorphism  $\phi_\alpha$  in the last corollary is injective.

(2)  $Z(G) \cong \text{Hom}(A/Q, k^\times)$ . For example if  $k = \mathbb{C}$  then  $Z(G) \cong A/Q \dots$

(3) The normalizer of  $U$  in  $G$  is  $B$ . Assuming  $|k| > 3$ , the normalizer of  $H$  in  $G$  is  $N$ .