Math 648 Midterm

Answer as many questions as you can! Make sure you state clearly any theorems from class that you use.

Part I. Definitions.

1. State the Krull-Schmidt theorem.

   If M is a left R-module of finite length, then M can be written as a direct sum $M = M_1 \oplus \cdots \oplus M_n$ for non-zero indecompsable submodules $M_1, \dots, M_n$. Moreover, given another such decomposition $M = M_1' \oplus \cdots \oplus M_{n'}'$, we have that $n = n'$ and there is a permutation $w \in S_n$ such that $M_i \cong M_{wi}'$.

Part II. True or False. Justify your answers briefly.

1. If $R = F[x, y]$ for a field F, then every submodule of a free R-module of finite rank is free.

   False. Take the regular module (which is free of rank 1) and let M be the ideal (x, y). Suppose it is free with some basis $x_1, \dots, x_n$. Since M is not a principle ideal, we must have $n > 1$. But then $x_1x_2 - x_2x_1 = 0$, contradicting linear independence.

2. Every short exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with C semisimple is split.

   False. For example take $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ where the first map is multiplication by 2. Since $\mathbb{Z}_2$ is simple its semisimple. But this does not split, since $\mathbb{Z}_2$ is not a subgroup of $\mathbb{Z}$ (the latter has no element of order 2).

3. In a ring R the set of nilpotent elements, i.e. the $x \in R$ such that $x^n = 0$ for some $n \gg 0$, is a two-sided ideal.

   False. It is true if R is commutative. But for instance take $R = M_2(\mathbb{C})$. There are non-zero nilpotent matrices, but not every matrix is nilpotent. But R is a simple ring so the only two-sided ideals are 0 and R itself.

4. If R is an integral domain and $0 \neq x \in R$ is an irreducible element then (x) is a maximal ideal.

   False. Let $R = F[x, y]$ for F a field. Clearly x is irreducible since it is of degree 1 and non-zero scalars are units. But $(x) \subset (x, y)$ so it is not a maximal ideal.

5. Let R be a ring, $V < W$ and $V' < W'$ be left R-modules such that $V \cong V'$ and $W/V \cong {}_RR \cong W'/V'$. Then, $W \cong W'$.

   True. The short exact sequence $0 \to V \to W \to R \to 0$ splits since R is free. Hence $W \cong V \oplus R$. Similarly $W' \cong V' \oplus R$. So $W \cong W'$ since $V \cong V'$.

Part III. Longer problems.

1. Let V and W be non-isomorphic irreducible left R-modules. Prove that the R-module $V \oplus W$ is cyclic.

   Solution. Take any $0 \neq v \in V$ and $0 \neq w \in W$. Let $x = v + w$. I claim $Rx = V \oplus W$. To prove this, it suffices by the Jordan-Holder theorem to show that Rx is not simple. Well suppose for a contradiction that Rx is simple. The first projection $V \oplus W \twoheadrightarrow V$ restricted to the submodule Rx defines a homomorphism $Rx \to V$ which is onto, hence an isomorphism since we are assuming that Rx is simple. Similarly the second projection restricts to an isomorphism $Rx \to W$. Hence $V \cong Rx \cong W$, contradiction.

2. Compute the invariant factors $d_1|d_2|d_3$ of the matrix

$$\begin{bmatrix} 2i & i & 2+i \\ i-1 & 1+i & 0 \\ 0 & 0 & 2+i \\ 1+i & -1 & 2+i \end{bmatrix}$$

working in the ring $\mathbb{Z}[i]$ of Gaussian integers.

Solution. Start off by doing some obvious row and column operations. You quickly get the matrix down to something like $\mathrm{diag}(1, 2+i, 3+i)$. But then $2+i$ doesn't divide $3+i$ so we're not done. This is the point when you have to do something difficult that is not a simple row/column operation. At this point it is easier to consider the J_i's: clearly $J_1 = (1)$, $J_2 = (2+i, 3+i) = (1)$ and $J_3 = ((2+i)(3+i)) = (5+5i)$. Hence the invariant factors are $1, 1, 5+5i$.

3. Let R be a commutative ring and P be a prime ideal.

(a) Write down the universal property that defines the localization of R at the prime ideal P.

(b) How many elements are there in the localization of the ring $R = \mathbb{Z}_{375}$ at the prime ideal $P = (5)$?

Solution. (a) Let $S = R - P$. Then the localization of R at P is a ring $S^{-1}R$ plus a ring homomorphism $i : R \to S^{-1}R$ such that $i(s)$ is a unit for every $s \in S$. Moreover, given another ring homomorphism $f : R \to T$ such that every element of $f(S)$ is a unit in T, there exists a unique $\bar{f} : S^{-1}R \to T$ such that $f = \bar{f} \circ i$.

(b) Let $S = \{\text{integers prime to 5 in } \mathbb{Z}_{375}\}$. Suppose that $f : \mathbb{Z}_{375} \to T$ is a ring homomorphism such that $f(S)$ consists of units. In particular, $f(3)$ is a unit, hence since $f(375) = 0 = f(125)f(3)$ we see that $f(125) = 0$. Hence f factors uniquely through the quotient $\mathbb{Z}_{125} = \mathbb{Z}_{375}/(125)$ to induce a homomorphism $\mathbb{Z}_{125} \to T$. Finally observe that every integer prime to 5 is a unit in $\mathbb{Z}_{125}$. This proves that $\mathbb{Z}_{125}$ together with the map sending n mod 375 to n mod 125 is the localization of $\mathbb{Z}_{375}$ at the prime ideal (5).

Hence the answer is 125.