Math 648 Midterm

Answer as many questions as you can! Make sure you state clearly any theorems from class that you use.

Part I. Definitions.

1. State the Krull-Schmidt theorem.

Part II. True or False. Justify your answers briefly.

1. If $R = F[x, y]$ for a field F, then every submodule of a free R-module of finite rank is free.
2. Every short exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with C semisimple is split.
3. In a ring R the set of nilpotent elements, i.e. the $x \in R$ such that $x^n = 0$ for some $n \gg 0$, is a two-sided ideal.
4. If R is an integral domain and $0 \neq x \in R$ is an irreducible element then (x) is a maximal ideal.
5. Let R be a ring, $V < W$ and $V' < W'$ be left R-modules such that $V \cong V'$ and $W/V \cong {}_R R \cong W'/V'$. Then, $W \cong W'$.

Part III. Longer problems.

1. Let V and W be non-isomorphic irreducible left R-modules. Prove that the R-module $V \oplus W$ is cyclic.
2. Compute the invariant factors $d_1 | d_2 | d_3$ of the matrix

$$\begin{bmatrix} 2i & i & 2+i \\ i-1 & 1+i & 0 \\ 0 & 0 & 2+i \\ 1+i & -1 & 2+i \end{bmatrix}$$

   working in the ring $\mathbb{Z}[i]$ of Gaussian integers.

3. Let R be a commutative ring and P be a prime ideal.

   (a) Write down the universal property that defines the localization of R at the prime ideal P.

   (b) How many elements are there in the localization of the ring $R = \mathbb{Z}_{375}$ at the prime ideal $P = (5)$?