

Math 647 Midterm

Answer as many questions as you can! Make sure you state clearly any theorems from class that you use.

Part I. Definitions.

1. In a category \mathcal{C} , what is the universal property satisfied by the *coproduct* $A \sqcup B$ of objects A and B ?

Part II. True or False. Justify your answers briefly.

1. All subgroups of a direct product $G \times H$ of groups G and H are of the form $G' \times H'$ for $G' \leq G$ and $H' \leq H$.
2. If P is a Sylow p -subgroup of a finite group G and $P \trianglelefteq N \trianglelefteq G$ for some N then $P \trianglelefteq G$.
3. $\text{Aut}(C_8) \cong C_4$.
4. A_5 has no subgroup of order 15.
5. There are 9 orbits for the action of A_5 on S_5 by conjugation.

Part III. Longer problems.

1. Suppose that V is a vector space over a field k with basis $e_1, \dots, e_n, f_1, \dots, f_n$. Let (\cdot, \cdot) denote the skew-symmetric bilinear form on V defined by $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{i,j}$ (i.e. 1 if $i = j$, 0 if $i \neq j$), for all $i, j = 1, \dots, n$. If U is a subspace of V with the property that $U \subseteq U^\perp$, prove that $\dim U \leq n$. Give an example of a subspace U of V such that $U = U^\perp$.
2. Prove that the symmetric group S_n is generated by the transposition (12) and the n -cycle $(12 \dots n)$. Is S_n generated by *any* pair of elements consisting of a transposition and an n -cycle?
3. Let G be a finite p -group and $\{1\} \neq N \trianglelefteq G$ be a normal subgroup. Prove that $N \cap Z(G) \neq \{1\}$, where $Z(G)$ denotes the *center* $\{g \in G \mid gh = hg \text{ for all } h \in G\}$ of the group G .
- 4* Let $|G| = p(p+1)$ where p is prime. Prove that G has either a normal subgroup of order p or a normal subgroup of order $(p+1)$.