

Math 649 Midfinal

Answer as many questions as you can! Make sure you state clearly any theorems from class that you use.

Part I. Definitions.

1. Give two different characterizations of the Jacobson radical $J(R)$ of a left Artinian ring R .
Solution.
(1) It is the smallest two-sided ideal J of R such that R/J is a semisimple left R -module.
(2) It is the largest nilpotent left ideal of R .

Part II. True or False. Justify your answers briefly.

1. For any commutative ring R , the R -algebras $R[x] \otimes_R R[x]$ and $R[x, y]$ are isomorphic.
True. An isomorphism maps $x \otimes 1$ to x and $1 \otimes x$ to y .
2. If R is a ring having no non-trivial two-sided ideals, then R is a division algebra.
False. For example $M_2(F)$ for F a field. Its a simple ring (e.g. because it is Morita equivalent to the field F), but there are non-zero 2×2 nilpotent matrices which are not invertible.
3. If R is a principal ideal domain, then R is Noetherian.
True. Take an ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of ideals in R . We need to show it stabilizes. Let $I = \bigcup_{n \geq 1} I_n$. It is also an ideal of R . Since R is a PID, $I = (a)$ for some $a \in R$. But then $a \in I_n$ for some n , and hence $I_n = I$ already. Hence $I_n = I_{n+1} = \dots$ and the chain has stabilized.
4. If R is a commutative Noetherian ring, then every $R/J(R)$ -module is semisimple.
False. Take for instance $R = \mathbb{Z}$. It is a PID so its Noetherian. Its Jacobson radical is the intersection of all maximal ideals (p) for p prime. That is an ideal of the form (n) where n is divisible by all primes. Hence it is (0) . But not every \mathbb{Z} -module is semisimple (e.g. $\mathbb{Z}/(4)$ is not).
5. If F is a field of characteristic $p > 0$ and G is a finite abelian p -group, then there is only one irreducible FG -module up to isomorphism.
True. Note G is a product of cyclic groups of order p^i for various i . So it suffices to prove the result just for G cyclic of order p^i . But then $FG \cong F[x]/(x^{p^i} - 1) = F[x]/(x - 1)^{p^i}$. This clearly has just one irreducible module by the structure theorem for modules over PIDs (there is just one prime in the ring $F[x]/(x - 1)^{p^i}$, namely, $(x - 1)$).

Part III. Longer problems.

1. Let G be a finite group and F be an algebraically closed field of characteristic $p \geq 0$.
 - (i) Prove that up to isomorphism, there are only finitely many irreducible FG -modules L_1, \dots, L_r .
 - (ii) Let $n_i = \dim_F L_i$, $i = 1, \dots, r$. Prove that $\sum_{i=1}^r n_i^2 \leq |G|$, with equality if and only if $p = 0$ or $p \nmid |G|$.
 - (iii) Is it true that the inequality $\sum_{i=1}^r n_i^2 \leq |G|$ holds even if F is not algebraically closed?

Solution.

Since G is finite, FG is an Artinian ring. Hence, its Jacobson radical $J(FG)$ acts as zero on any irreducible FG -module, i.e. any irreducible FG -module is a lift of an irreducible $FG/J(FG)$ -module. The latter is a semisimple algebra over an algebraically closed field. Hence by (souped-up) Wedderburn theorem, $FG/J(FG) \cong M_{n_1}(F) \times \cdots \times M_{n_r}(F)$ where r is the number of isomorphism classes of irreducible module FG -modules (FINITE, PROVING (i)!) and n_1, \dots, n_r are the dimensions of corresponding irreducible representations. Hence $n_1^2 + \cdots + n_r^2 = \dim FG - \dim J(FG)$. So $n_1^2 + \cdots + n_r^2 \leq |G|$, with equality if and only if $J(FG) = 0$, i.e. FG is a semisimple algebra. By Maschke's theorem (and its converse, stated but not proved in class) FG is a semisimple algebra if and only if $p = 0$ or $p \nmid |G|$. This gives (ii). Finally for (iii), this statement is false. Consider for example the group algebra $\mathbb{R}Q_3$. There are five irreps, of \mathbb{R} -dimensions 1,1,1,1,4 (the latter being the quaternions...). The squares add up to more than 8 the last time I checked.

2. Let $f : V \rightarrow V$ be an endomorphism of an n -dimensional vector space over a field F . Let $x^n - c_1x^{n-1} + c_2x^{n-2} - \cdots + (-1)^n c_n$ be the characteristic polynomial of the linear map f . For each $k = 1, \dots, n$, prove that $c_k = \text{tr}(\wedge^k f)$, where $\wedge^k f : \wedge^k V \rightarrow \wedge^k V$ is the linear map with $(\wedge^k f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k)$ for all $v_1, \dots, v_k \in V$.

Solution. Let \bar{F} be the algebraic closure of F . Replacing V by $\bar{F} \otimes_F V$ and f by $1 \otimes f$ does not affect either the characteristic polynomial or the traces of the linear maps $\wedge^k f$ (because $1 \otimes \wedge^k f = \wedge^k(1 \otimes f)$). So we may assume to do this problem that the field F is already algebraically closed.

In that case, there exists a basis v_1, \dots, v_n with respect to which the matrix of f is upper triangular, with eigenvalues $\lambda_1, \dots, \lambda_n$ down the diagonal. The characteristic polynomial is $\prod_{i=1}^n (x - \lambda_i)$. So the x^{n-i} -coefficient is $(-1)^{n-i} \sum_{1 \leq j_1 < \cdots < j_i \leq n} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i}$ (the i th elementary symmetric function in the λ 's).

Now consider $\wedge^i f$. A basis for $\wedge^i V$ is given by the $v_{j_1} \wedge \cdots \wedge v_{j_i}$ for $1 \leq j_1 < \cdots < j_i \leq n$. The matrix of $\wedge^i f$ in this basis suitably ordered (so $v_1 \wedge \cdots \wedge v_i$ appears first, ...) is again upper triangular, with $\lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i}$ appearing down the diagonal. Hence the trace of $\wedge^i f$ is again the i th elementary symmetric function in the λ 's. This is what we needed.

3. Let $V = \{1, x, y, z\}$ be the Klein 4-group and $C = \{1, a, a^2\}$ be the cyclic group of order 3. Let $H = V \rtimes C$ be their semidirect product constructed so that $axa^{-1} = y$ in H . Compute the character table of the group H . Hence, or otherwise, list all the normal subgroups of H .

Solution. First we need the conjugacy classes. Note $H = \{1, a, a^2, x, xa, xa^2, y, ya, ya^2, z, za, za^2\}$. By inspection, the conjugacy classes are $C_1 = \{1\}$, $C_2 = \{x, y, z\}$, $C_3 = \{a, xa, ya, za\}$ and $C_4 = \{a^2, xa^2, ya^2, za^2\}$. Hence there are four irreducible characters, of degrees 1, 1, 1, 3 since their squared degrees add up to 12. The three degree one ones must come from lifts of the three irreducible characters of C . The remaining one can be worked out by the orthogonality relations.

	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

(Here ω is a primitive cube root of 1.)

The normal subgroups are now easy: the lattice of normal subgroups is generated by the $\ker \chi_i$'s. From the character table, the only normal subgroup other than $\{1\}$ and H is $V = C_1 \cup C_2$.

By the way: of course H is the alternating group A_4 . A better question would be to find the character table of $T = C \rtimes V$.