Math 649 Midfinal

Answer as many questions as you can! Make sure you state clearly any theorems from class that you use.

Part I. Definitions.

1. Give two different characterizations of the Jacobson radical $J(R)$ of a left Artinian ring R.

Part II. True or False. Justify your answers briefly.

1. For any commutative ring R, the R-algebras $R[x] \otimes_R R[x]$ and $R[x, y]$ are isomorphic.
2. If R is a ring having no non-trivial two-sided ideals, then R is a division algebra.
3. If R is a principal ideal domain, then R is Noetherian.
4. If R is a commutative Noetherian ring, then every $R/J(R)$-module is semisimple.
5. If F is a field of characteristic $p > 0$ and G is a finite abelian p-group, then there is only one irreducible FG-module up to isomorphism.

Part III. Longer problems.

1. Let G be a finite group and F be an algebraically closed field of characteristic $p \geq 0$.
   (i) Prove that up to isomorphism, there are only finitely many irreducible FG-modules $L_1, \dots, L_r$.
   (ii) Let $n_i = \dim_F L_i$, $i = 1, \dots, r$. Prove that $\sum_{i=1}^r n_i^2 \leq |G|$, with equality if and only if $p = 0$ or $p \nmid |G|$.
   (iii) Is it true that the inequality $\sum_{i=1}^r n_i^2 \leq |G|$ holds even if F is not algebraically closed?
2. Let $f : V \to V$ be an endomorphism of an n-dimensional vector space over a field F. Let $x^n - c_1 x^{n-1} + c_2 x^{n-2} - \cdots + (-1)^n c_n$ be the characteristic polynomial of the linear map f. For each $k = 1, \dots, n$, prove that $c_k = \operatorname{tr}(\bigwedge^k f)$, where $\bigwedge^k f : \bigwedge^k V \to \bigwedge^k V$ is the linear map with $(\bigwedge^k f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k)$ for all $v_1, \dots, v_k \in V$.
3. Let $V = \{1, x, y, z\}$ be the Klein 4-group and $C = \{1, a, a^2\}$ be the cyclic group of order 3. Let $H = V \rtimes C$ be their semidirect product constructed so that $axa^{-1} = y$ in H. Compute the character table of the group H. Hence, or otherwise, list all the normal subgroups of H.