

# Exercises on chapter 5

1. Let  $E \subseteq F \subseteq K$  be field extensions. Prove that  $\text{trdeg}_E(K) = \text{trdeg}_F(K) + \text{trdeg}_E(F)$ .

Let  $a_1, \dots, a_n$  be a transcendence base for  $F$  over  $E$ . Let  $b_1, \dots, b_m$  be a transcendence base for  $K$  over  $F$ . I claim that  $a_1, \dots, a_n, b_1, \dots, b_m$  is a transcendence base for  $K$  over  $E$ .

First they are algebraically independent. Suppose that  $f(x_1, \dots, x_n, y_1, \dots, y_m)$  is a polynomial with coefficients in  $E$  such that  $f(a_1, \dots, a_n, b_1, \dots, b_m) = 0$ . Then  $g(y_1, \dots, y_m) = f(a_1, \dots, a_n, y_1, \dots, y_m)$  is a polynomial with coefficients in  $F$  which is zero when evaluated at  $b_1, \dots, b_m$ . Since  $b_1, \dots, b_m$  are algebraically independent over  $F$  this implies that  $g = 0$ . Now consider the  $y_1^{n_1} \cdots y_m^{n_m}$ -coefficient of  $g$  (which we've shown is zero). It is the  $y_1^{n_1} \cdots y_m^{n_m}$ -coefficient of  $f$  evaluated at  $x_i = a_i$ . Since the  $a_i$  are algebraically independent over  $E$  this implies that all the coefficients of  $f$  are already zero. Hence  $f = 0$ .

Now we just need to show that any  $\alpha \in K$  is algebraic over  $E(a_1, \dots, a_n, b_1, \dots, b_m)$ . Well,  $K$  is algebraic over  $F(b_1, \dots, b_m)$ . Hence there exists a non-zero polynomial  $f(x) \in F[b_1, \dots, b_m][x]$  with  $\alpha$  as a root. The coefficients of  $f(x)$  involve only finitely many elements  $s_1, \dots, s_p$  of  $F$ . Hence  $\alpha$  is algebraic over  $E(s_1, \dots, s_p, b_1, \dots, b_m)$ . In turn,  $E(s_1, \dots, s_p, b_1, \dots, b_m)$  is algebraic over  $E(a_1, \dots, a_n, b_1, \dots, b_m)$ , because each  $s_i$  is algebraic over  $E(a_1, \dots, a_n)$ . Hence (by transitivity of algebraic extensions)  $\alpha$  is algebraic over  $E(a_1, \dots, a_n, b_1, \dots, b_m)$ .

2. Let  $F$  be the field  $\mathbb{Q}(x_1, \dots, x_n)$  (rational functions in  $n$  indeterminates over  $\mathbb{Q}$ ). Define the *elementary symmetric functions*

$$e_i = \sum_{1 \leq j_1 < \dots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}.$$

For instance  $e_1 = x_1 + x_2 + \dots + x_n$  and  $e_n = x_1 x_2 \cdots x_n$ . Let  $E$  be the field  $\mathbb{Q}(e_1, \dots, e_n)$ , so  $\mathbb{Q} \subset E \subset F$  is a tower of field extensions.

- (i) Prove that  $F$  is an algebraic extension of  $E$ .
- (ii) Prove that  $e_1, \dots, e_n$  is a transcendence base for  $F$  over  $\mathbb{Q}$ . (So is  $x_1, \dots, x_n$ , of course).
- (i) We just need to show each  $x_i$  is algebraic over  $E$ . Consider the polynomial  $(x-x_1)(x-x_2) \cdots (x-x_n)$ . When you expand, the coefficients of  $x^i$ 's are precisely the elementary symmetric functions  $e_1, \dots, e_n$  (plus or minus). So this polynomial lies in  $E[x]$ . Each  $x_i$  is a root. So each  $x_i$  is algebraic over  $E$ .
- (ii) Note  $x_1, \dots, x_n$  is a transcendence base by definitions, so the transcendence degree of  $F$  over  $\mathbb{Q}$  is  $n$ . Since there are  $n$  elements in the set  $e_1, \dots, e_n$ , we therefore just need to show that  $F$  is algebraic over  $\mathbb{Q}(e_1, \dots, e_n)$  (i.e. they span...). We did that in (i).

3. Continue with the notation from question 2. To do this question, you'll need the review some facts from the fundamental theorem of Galois theory from §4.2 of Rotman.

- (i) Prove that the field  $F$  is the splitting field of the polynomial  $f(x) = (x-x_1)(x-x_2) \cdots (x-x_n) \in E[x]$ .
- (ii) Let  $G = \text{Gal}(F/E)$  be the Galois group of the extension  $E \subset F$ , i.e. the group of all field automorphisms of  $F$  that fix the subfield  $E$  pointwise. By considering the action of  $G$  on the roots  $x_1, \dots, x_n$ , prove that  $|G| \leq n!$ .

(iii) Prove that  $G$  is equal to the symmetric group  $S_n$  acting on  $F$  by permuting the variables  $x_1, \dots, x_n$ .

(iv) Prove that “symmetric functions are rational functions in the elementary symmetric functions”. This sentence means: the invariant subfield  $F^G = \{\alpha \in F \mid g\alpha = \alpha \text{ for all } g \in G\}$  of  $G$  acting on  $F$  is equal to  $E$ .

(v) Prove that  $[F : E] = n!$ .

(vi) In the case  $n = 2$  prove that  $\{1, x_1\}$  forms a basis for  $F$  as an  $E$ -vector space.

(vii) In the case  $n = 3$  prove that  $\{1, x_1, x_1^2, x_2, x_1x_2, x_1^2x_2\}$  forms a basis for  $F$  as an  $E$ -vector space.

(viii) Can you find a basis for  $F$  as an  $E$ -vector space in general?

(i) Well the polynomial does split over  $F$ . So we just need to show that  $F = E(x_1, \dots, x_n)$ . This is clear.

(ii) The Galois group acts faithfully on the set of roots of the polynomial  $f(x)$ . Hence  $G$  is a subgroup of  $S_n$ . Hence,  $|G| \leq n!$ .

(iii) We just need to show that each  $g \in S_n$  does define a unique field automorphism of  $F$  sending  $x_i$  to  $x_{g(i)}$ . Then  $S_n \leq G$  hence they're equal by (ii). To do this, start from a map  $\mathbb{Q}[x_1, \dots, x_n] \rightarrow F$  mapping  $x_i$  to  $x_{g(i)}$  (this is fine because polynomial algebra is free). Then use the universal property of the field of fractions to see that it extends to a field automorphism of  $F$ .

(iv) Well,  $E \subset F$  is a Galois extension, so  $E$  is the fixed subfield of the Galois group by the fundamental theorem of Galois theory.

(v) And  $[F : F^G] = |G|$  hence  $[F : E] = n!$  by the fundamental theorem of Galois theory.

(vi) By dimension, we just need to show that they span. We need to write any monomial  $x_1^i x_2^j$  as an  $E$ -linear combination of 1 and  $x_1$ . Any symmetric polynomial in  $x_1$  and  $x_2$  is certainly in  $E$  by (iv). Then

$$x_1^{n+1} = (x_1^n + x_1^{n-1}x_2 + \dots + x_1x_2^{n-1} + x_2^n)x_1(x_1^n x_2 + \dots - x_1x_2^n)$$

which is (symmetric poly) times  $x_1$  minus (symmetric poly). So  $x_1^{n+1}$  is a linear combination for all  $n \geq 0$ . From this you get all  $x_1^i x_2^j$  for  $i \geq j$ . But  $x_1^i x_2^j + x_1^j x_2^i$  is symmetric so you now get all for  $i \leq j$  too...

(vii) See (viii). I didn't try to prove it yet, but it should be something silly like (vi).

(viii) You can take the following set of  $n!$  vectors:  $\{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid 0 \leq \alpha_i \leq n - i\}$ .

4. This is a continuation of questions 2 and 3... Let  $S = \mathbb{Q}[x_1, \dots, x_n]$  be the polynomial ring. Note the monomials  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$  for all tuples  $\alpha = (\alpha_1, \dots, \alpha_n)$  with each  $\alpha_i \in \mathbb{N} = \{0, 1, 2, \dots\}$  forms a basis for the  $\mathbb{Q}$ -vector space  $S$ . So any  $f(x) \in S$  can be written as  $f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \in S$  for  $c_\alpha \in \mathbb{Q}$ .

(i) The symmetric group  $G = S_n$  acts on  $S$  by algebra automorphisms permuting the variables:  $gx_i = x_{g(i)}$ . Let  $S^G = \{x \in S \mid gx = x \text{ for all } g \in G\}$  be the ring of symmetric polynomials. Let  $e_1, \dots, e_n$  be the elementary symmetric functions as defined above. Prove that the subring  $R = \mathbb{Q}[e_1, \dots, e_n]$  of  $S$  is contained in  $S^G$  and that  $R \cong S^G$  as  $\mathbb{Q}$ -algebras.

(ii) Define a total ordering on the set  $\mathbb{N}^n$  by declaring that  $\alpha < \beta$  if  $\alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}$  and  $\alpha_i < \beta_i$  for some  $i = 1, \dots, n$ . (This is the *lexicographical ordering*.) Define the *degree* of  $f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \in S$  to be the biggest  $\alpha \in \mathbb{N}^n$  such that  $c_\alpha \neq 0$ . Prove that the degree  $\beta$  of any symmetric polynomial satisfies  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$ .

(iii) If  $e_1, \dots, e_n$  are the elementary symmetric functions, prove that the degree of  $e_i$  is  $(1, \dots, 1, 0, \dots, 0)$  ( $i$  1's then  $(n - i)$  0's).

(iv) Let  $(\gamma_1, \dots, \gamma_n) = (\beta_1 - \beta_2, \beta_2 - \beta_3, \dots, \beta_{n-1} - \beta_n, \beta_n)$ . Prove that if  $g(x_1, \dots, x_n) = x_1^{\gamma_1} \dots x_n^{\gamma_n}$  then  $g(e_1, \dots, e_n)$  is symmetric of degree  $\beta$ .

(v) Prove by induction on degree that every symmetric polynomial  $f(x) \in S^G$  is a polynomial in  $e_1, \dots, e_n$ , i.e.  $S^G = R$ . This is the *fundamental theorem of symmetric polynomials*.

(i) Since each  $e_i$  is  $G$ -invariant, it is clear that  $R \subseteq S^G$ . Also we showed last week that  $e_1, \dots, e_n$  was a transcendence base for the field extension  $\mathbb{Q} \subset \mathbb{Q}(x_1, \dots, x_n)$ , in particular  $e_1, \dots, e_n$  were algebraically independent. Hence  $\mathbb{Q}[e_1, \dots, e_n] \cong \mathbb{Q}[x_1, \dots, x_n]$ .

(ii) If  $x^\alpha$  appears with non-zero coefficient in  $f(x)$ , choose  $g \in S_n$  so that  $\alpha_{g1} \geq \alpha_{g2} \geq \dots \geq \alpha_{gn}$ . Then  $x^\beta = x_1^{\alpha_{g1}} \dots x_n^{\alpha_{gn}}$  also appears with non-zero coefficient, and  $\beta \geq \alpha$  in the lexicographical ordering.

(iii) The leading term of  $e_i$  is  $x_1 x_2 \dots x_i$  of degree  $(1, 1, \dots, 1, 0, \dots, 0)$ .

(iv) Think about the biggest monomial when  $g(e_1, \dots, e_n)$  is expanded. By (iii) it arises from the term  $x_1^{\gamma_1} (x_1 x_2)^{\gamma_2} \dots (x_1 \dots x_{n-1})^{\gamma_{n-1}} (x_1 \dots x_n)^{\gamma_n} = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$ . So its symmetric of degree  $\beta$ .

(v) Proceed by induction on degree. For the induction step, let  $f(x) \in S^G$ . Pick  $\beta$  maximal so that  $x^\beta$  occurs in  $f(x)$  with non-zero coefficient. Note by (ii) that  $\beta_1 \geq \dots \geq \beta_n$ . Define  $g(x_1, \dots, x_n)$  as in (iv). Then  $f(x) - c_\beta g(e_1, \dots, e_n)$  is also in  $S^G$  and it is of strictly smaller degree, so by induction lies in  $R$ . Since  $g(e_1, \dots, e_n)$  lies in  $R$  too, we deduce that  $f(x) \in R$ .

5. Let  $R \subseteq S$  be an integral extension and assume that  $S$  is an integral domain. Prove that  $R$  is a field if and only if  $S$  is a field.

See Lemma 6.1.4 from the notes.

6. Let  $S$  be an integral domain and  $R \subset S$  be a unital subring. Let  $E \subset F$  be the corresponding fields of fractions. Assume that  $S$  is a free  $R$ -module of rank  $r$ . Prove that  $[F : E] = r$  too.

Let  $x_1, \dots, x_r$  be a basis for  $S$  as a free  $R$ -module. There are natural embeddings of  $R$  into  $E$  and  $S$  into  $F$ . So we can view  $x_1, \dots, x_r$  as elements of  $F$  too. I claim that they form a basis for  $F$  as an  $E$ -vector space, hence  $[F : E] = r$ .

To see this, suppose first that  $a_1 x_1 + \dots + a_r x_r = 0$  for  $a_1, \dots, a_r \in E$ . Each  $a_i$  is a fraction of the form  $p_i/q_i$  for  $p_i, q_i \in R$ . Multiplying by  $0 \neq q_1 \dots q_r = q$ , we deduce that  $q a_1 x_1 + \dots + q a_r x_r = 0$ . Each  $q a_i$  lies in  $R$  and the  $x_i$  were linearly independent over  $R$ , hence  $q a_i = 0$  for each  $i$ . Since  $R$  is an integral domain that implies that  $a_i = 0$  for all  $i$ . Hence linear independence.

Now to see that they span, it is obvious that any element  $\alpha$  of  $F$  can be written as

$$\frac{a_1 x_1 + \dots + a_r x_r}{b_1 x_1 + \dots + b_r x_r}$$

for  $a_i, b_i \in R$  (and the denominator non-zero of course). But we're in an integral extension, so  $x = b_1 x_1 + \dots + b_r x_r$  is integral over  $R$ . Therefore  $x^n + c_1 x^{n-1} + \dots + c_n = 0$  for some  $c_i \in R$  with  $x_n \neq 0$ . Hence  $x^{-1} = -c_n^{-1}(x^{n-1} + c_1 x^{n-2} + \dots + c_{n-1})$ . Each positive power of  $x$  lies in  $S$  so is an  $R$ -linear combination of  $x_1, \dots, x_n$ . We deduce that  $x^{-1}$  is an  $E$ -linear combination of  $x_1, \dots, x_n$  too. Hence  $\alpha$  is an  $E$ -linear combination of  $x_1, \dots, x_n$ , and they span.

7. Let  $R$  and  $S$  be as in question 4.

(i) Prove that  $S$  is integral over  $R$ .

(ii) It is known that  $S$  is a free  $R$ -module (I decided it would take too long to get you to prove this fact in this exercise, so you should believe it without proof). What is the rank of  $S$  as a free  $R$ -module? (Hint: look at the earlier problems!).

(iii) Let  $I$  be the ideal of  $S$  generated by the elementary symmetric functions  $e_1, \dots, e_n$ , and let  $C = S/I$  be the quotient algebra, i.e.  $C = \mathbb{Q}[x_1, \dots, x_n]/\langle e_1, \dots, e_n \rangle$ . If  $b_1, \dots, b_r$  is a basis for  $S$  as a free  $R$ -module, prove that their images  $\bar{b}_1, \dots, \bar{b}_r$  in  $S/I$  form a  $\mathbb{Q}$ -basis for  $C$ . Deduce that  $C$  is a finite dimensional commutative algebra of dimension  $n!$ .

(iv) Compute the Jacobson radical of  $C$  and hence classify all irreducible  $C$ -modules up to isomorphism.

(Remark. Generalizing question 7, it can be shown that the ring  $C_{\mathbb{Z}} = \mathbb{Z}[x_1, \dots, x_n]/\langle e_1, \dots, e_n \rangle$  is a free  $\mathbb{Z}$ -module on basis  $\{x_1^{a_1} \dots x_n^{a_n} \mid 0 \leq a_i \leq n - i\}$ . This remarkable ring  $C_{\mathbb{Z}}$  is isomorphic to the cohomology ring  $H^*(F_n)$  of the flag manifold of an  $n$ -dimensional complex vector space, with multiplication being cup product...)

(i) Clearly  $S = R[x_1, \dots, x_n]$ . It suffices to show each  $x_i$  is integral over  $R$ . But  $x_i$  is a root of the monic polynomial  $(x - x_1) \cdots (x - x_n) \in R[x]$ .

(ii) By question 6, given that  $S$  is a free  $R$ -module, its rank is the same as the degree  $[F : E]$  of the corresponding fields of fractions which you showed using Galois theory last week was  $n!$ .

(iii) Let  $b_1, \dots, b_r$  be a basis for  $S$  as a free  $R$ -module (where we know that  $r = n!$ ). Take  $x \in S$ . It can be written as  $a_1 b_1 + \cdots + a_r b_r$  for  $a_i \in R$ . Hence  $\bar{x}$ , the image of  $x$  in  $S/I$ , is  $\bar{a}_1 \bar{b}_1 + \cdots + \bar{a}_r \bar{b}_r$ . Now everything in  $R$  looks like a constant plus stuff involving  $e_i$ 's, so things in  $R$  map to scalars in  $S/I$ . Hence  $\bar{x}$  is a  $\mathbb{Q}$ -linear combination of  $\bar{b}_1, \dots, \bar{b}_r$ . Thus they span  $S/I$  as a vector space. This shows that  $\dim_{\mathbb{Q}} C \leq n!$ .

Now things get a little harder. The trick is to proceed by induction on degree. Of course,  $S = \bigoplus_{d \geq 0} S_d$  where  $S_d$  is the span of all monomials of degree  $d$  (e.g.  $S_0$  is just the scalars,  $S_1$  is spanned by the  $x_i$ ,  $S_2$  is spanned by the  $x_i x_j, \dots$ ). In fact this makes  $S$  into what is called a *graded algebra*, since if you multiply  $S_d$  by  $S_e$  you land up in  $S_{d+e}$ . (Other examples of graded algebras:  $T(V)$ ,  $S(V)$ ,  $\bigwedge(V)$  for a vector space  $V$ .) I call a polynomial  $f(x) \in S$  *homogeneous of degree  $d$*  if  $d(x) \in S_d$ . The quotient  $C$  is a quotient of  $S$  by an ideal generated by homogeneous elements, so  $C$  inherits a grading from  $S$  with  $C_d$  being the image of  $S_d$  under the quotient map  $S \rightarrow C$ .

Okay, with that notation, let  $b_1, \dots, b_r$  be homogeneous elements of  $S$  such that their images in  $S/I = C$  form a basis for  $C$  as a vector space. Let  $T$  be the  $R$ -span of  $b_1, \dots, b_r$ . Note that  $T$ , being generated by homogeneous elements, is also graded:  $T = \bigoplus_{d \geq 0} T_d$  where  $T_d = T \cap S_d$ . I'll show by induction on  $d$  that  $T_d = S_d$ . This completes the proof, for then  $T = S$  hence  $b_1, \dots, b_r$  span  $S$  as an  $R$ -module, hence (as  $R$  is commutative so has IBN)  $\dim_{\mathbb{Q}} C \geq n!$ .

Observe first as  $I_0 = 0$  that we certainly have that  $T_0 = S_0$ . This starts the induction. Now suppose we've shown  $T_0 = S_0, \dots, T_{d-1} = S_{d-1}$  and consider  $f(x) \in S_d$ . We can write

$$f(x) = \sum_i a_i b_i + \sum_j c_j d_j$$

where all of  $a_i, b_i, c_j, d_j$  are homogeneous polynomials in  $S$ ,  $a_i \in R$  and all the  $d_j$  belong to  $R$  and are of degree  $> 0$  (so that all  $c_j d_j$  belong to  $I$ ). By induction on degree, all  $c_j$ 's are  $R$ -linear combinations of the  $b_i$ 's. Hence  $f(x)$  is too, i.e.  $f(x) \in T_d$ . We're done.

(iv) Note that  $S = \bigoplus_{d \geq 0} S_d$  where  $S_d$  is the span of all monomials of degree  $d$ . Each  $e_i$  lies in  $S_i$ , so  $I$  is a homogeneous ideal, i.e.  $I = \bigoplus_{d \geq 0} I_d$  where  $I_d = I \cap S_d$ . Therefore  $S/I = \bigoplus_{d \geq 0} S_d/I_d$ . It is finite dimensional so  $S_d = I_d$  for  $d \gg 0$ . All this is compatible with multiplication (the fancy word is: its a *graded algebra*): if  $x \in S_d/I_d$  and  $y \in S_e/I_e$  then  $xy \in S_{d+e}/I_{d+e}$ . Hence  $\bigoplus_{d \geq 0} S_d/I_d$  is a nilpotent ideal. The quotient by this ideal is  $\mathbb{Q}$ , a semisimple algebra. Hence it is the Jacobson radical.

Since the Jacobson radical acts as zero on any semisimple module, the irreducible  $C$ -modules are the same as the irreducible  $C/J(C) = \mathbb{Q}$ -modules. So there's just one irreducible module up to isomorphism, namely,  $\mathbb{Q}$  with each  $\bar{x}_i$  acting as zero.

8. In all the remaining questions,  $F$  is an algebraically closed field and all algebras are commutative  $F$ -algebras.

(a) Prove that any prime ideal in the algebra  $F[x_1, \dots, x_n]$  is a radical ideal.

(b) For any ideal  $I$  of  $F[x_1, \dots, x_n]$ , prove that

$$\sqrt{I} = \bigcap_{I \subseteq J \text{ max}} J$$

where the intersection is over all maximal ideals  $J$  containing  $I$ . (Hint:  $\sqrt{I} = I(V(I))$ .)

(c) For any affine algebra  $A$  prove that its Jacobson radical  $J(A)$  is  $(0)$ .

(a) If  $I$  is a prime ideal, the quotient  $F[x_1, \dots, x_n]/I$  is an integral domain. Hence it has no non-zero zero divisors. Hence it certainly has no non-zero nilpotent elements. If  $a^n \in I$ , the image of  $a$  in the quotient ring is nilpotent, hence zero, hence already  $a \in I$ . This shows that  $I$  is a radical ideal.

(b) Note that  $\alpha \in V(I)$  if and only if  $\ker ev_\alpha \supseteq I$  (tautology!). By definition,

$$\begin{aligned} I(V(I)) &= \{f \in k[x_1, \dots, x_n] \mid f(\alpha) = 0 \text{ for all } \alpha \in V(I)\} \\ &= \{f \mid f \in \ker ev_\alpha \text{ for all } \alpha \in V(I)\} \\ &= \bigcap_{\alpha \in V(I)} \ker ev_\alpha. \end{aligned}$$

By the first remark, the right hand side is  $\bigcap_J J$  where the intersection is over all maximal ideals (a.k.a.  $\ker ev_\alpha$ 's by the weak Nullstellensatz) containing  $I$ .

(c) Say  $A = k[x_1, \dots, x_n]/I$  for  $I$  a radical ideal. By (b), the intersection of all maximal ideals of  $k[x_1, \dots, x_n]$  containing  $I$  is equal to  $I$ . Hence by the correspondence theorem, the intersection of all maximal ideals of  $A$  is equal to 0. Hence  $J(A) = (0)$ .

9. (a) Let  $X$  be a topological space and let  $S$  be a subset with the subspace topology. Prove that  $S$  is an irreducible topological space if and only if its closure  $\bar{S}$  in  $X$  is.

(b) Let  $f : X \rightarrow Y$  be a continuous map between topological spaces. Let  $S$  be an irreducible subset of  $X$  (i.e.  $S$  is irreducible in the subspace topology). Prove that  $f(S)$  is an irreducible subset of  $Y$ .

(c) Let  $A$  and  $B$  be affine algebras and  $f : A \rightarrow B$  be an algebra homomorphism. Let  $I$  be a prime ideal in  $B$ . Prove that  $f^{-1}(I)$  is a prime ideal in  $A$ . What has this got to do with (b)?

(d) Let  $I$  be a radical ideal in an affine algebra  $A$ . Prove that  $I$  can be expressed as an intersection of finitely many prime ideals.

(a) Remember closed sets of  $S$  are intersections of closed sets in  $X$  with  $S$ . So saying that  $S$  is irreducible means that if  $S \subseteq P \cup Q$  for closed sets  $P$  and  $Q$  in  $X$ , then either  $S \subseteq P$  or  $S \subseteq Q$ . If that's so for  $S$ , it's also so for  $\bar{S}$  by definition of closure. So  $S$  irreducible implies  $\bar{S}$  irreducible.

Conversely if  $\bar{S}$  is irreducible and  $S \subseteq P \cup Q$ , then  $\bar{S} \subseteq P \cup Q$  hence either  $\bar{S} \subseteq P$  or  $Q$ , hence  $S \subseteq P$  or  $Q$ .

(b) Suppose  $f(S)$  is reducible, i.e.  $f(S) \subseteq P \cup Q$  for closed subsets  $P$  and  $Q$  of  $Y$  neither of which contains  $f(S)$ . Then  $S \subseteq f^{-1}(P \cup Q) = f^{-1}(P) \cup f^{-1}(Q)$ . These are both closed. Hence as  $S$  is irreducible in  $X$ , we have without loss of generality that  $S \subseteq f^{-1}(P)$ . Hence  $f(S) \subseteq P$ . Contradiction.

(c) Let  $P$  be a prime ideal of  $B$ . Say  $ab \in f^{-1}(P)$ . Then  $f(a)f(b) \in P$ . Hence without loss of generality  $f(a) \in P$ . Hence  $a \in f^{-1}(P)$ .

This is actually a special case of (b), if you let  $Y$  and  $X$  be the affine varieties corresponding to  $A$  and  $B$  and consider the induced map  $X \rightarrow Y$  induced by the algebra homomorphism  $A \rightarrow B$ , remembering that prime ideals correspond to closed irreducible subsets.

(d) Let  $X$  be the affine variety corresponding to  $A$ . So  $V(I)$  is a closed set in  $X$ . Hence  $V(I)$  can be written as a union  $C_1 \cup \dots \cup C_n$  of irreducible closed subsets of  $X$ . Hence, as  $I$  is a radical ideal,  $I = I(C_1) \cap \dots \cap I(C_n)$ . Each  $I(C_i)$  is a prime ideal since each  $C_i$  is irreducible.

10. Let  $A$  and  $B$  be finitely generated algebras. (Commutative and over  $F$ , algebraically closed, of course.) I call a ring *reduced* if it has no non-zero nilpotent elements.

(a) If  $A$  and  $B$  are both reduced, prove that  $A \otimes_F B$  is reduced too.

(b) If  $A$  and  $B$  are both integral domains, prove that  $A \otimes_F B$  is an integral domain. (This is needed in class to show that the product of two irreducible affine varieties is irreducible.)

(c) Is  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$  an integral domain?

(a) If you look carefully, this is proved in Example 6.4.1(3) in the notes. Here's a direct proof without thinking about affine varieties. Let  $\sum_{i=1}^n a_i \otimes b_i$  be a nilpotent element of  $A \otimes B$ . We may assume the  $b_i$  are linearly independent. For any homomorphism  $h : A \rightarrow F$ , we have that  $h \otimes \text{id}$  is a homomorphism  $A \otimes B \rightarrow B$ . So  $\sum h(a_i)b_i$  is a nilpotent element of  $B$ , so it must be zero since  $B$  is reduced. Since the  $b_i$  are linearly independent, this means that all  $h(a_i) = 0$ . This was for all  $h$ ,

hence all the  $a_i$  lie in the intersection of all maximal ideals of  $A$ , which is zero by the Nullstellensatz. Hence our element was zero, and  $A \otimes B$  is reduced.

(b) Let  $f, g \in A \otimes B$  have  $fg = 0$ . Write  $f = \sum_i a_i \otimes b_i$  and  $g = \sum_j c_j \otimes d_j$ , the sets  $\{b_i\}$  and  $\{d_j\}$  being linearly independent. An argument like for (a) using that  $B$  is an integral domain then shows that  $a_i c_j = 0$  for all  $i, j$ . If  $a_i = 0$  for all  $i$  then  $f = 0$  and we're done. Else, some  $a_i \neq 0$ , whence all  $c_j = 0$  as  $A$  is an integral domain. Hence  $g = 0$ .

(c) NO! I think  $1 \otimes 1 + i \otimes i$  times  $1 \otimes 1 - i \otimes i$  is zero.

11. Describe all irreducible Hausdorff topological spaces.

(a) Points.

12. Let  $A$  and  $B$  be affine algebras. Prove that the maximal ideals of  $A \otimes_F B$  are all of the form  $A \otimes I + J \otimes B$  for  $I$  a maximal ideal of  $B$  and  $J$  a maximal ideal of  $A$ . (This is needed in class to show that the product  $X \times Y$  of two affine varieties is an affine variety with coordinate algebra  $F[X] \otimes_F F[Y]$ .) (Hint: it maybe easier to think in terms of irreducible modules.)

Since  $A \otimes_F B$  is a finitely generated  $F$ -algebra (which follows because a tensor product of  $F[x_1, \dots, x_n]$  and  $F[y_1, \dots, y_m]$  is isomorphic to  $F[x_1, \dots, x_n, y_1, \dots, y_m]$ ...) we know by the Nullstellensatz that all its irreducible modules are one dimensional. So if  $M$  is an irreducible module, its restriction to the subalgebra  $A = A \otimes 1$  is an irreducible  $A$ -module, say  $L_1$ , and its restriction to  $B = 1 \otimes B$  is an irreducible  $B$ -module, say  $L_2$ . Hence  $M \cong L_1 \otimes L_2$  (where the right hand side is the 'obvious'  $A \otimes B$ -module with action  $(a \otimes b)(l \otimes m) = al \otimes bm$ ). Conversely, all such tensor products are irreducible  $A \otimes B$ -modules. This classifies all irreducible  $A \otimes B$ -modules. To deduce the result about ideals, we just have to compute the annihilator of  $L_1 \otimes L_2$ . Clearly, if  $I$  is the annihilator of  $L_1$  in  $A$  and  $J$  is the annihilator of  $L_2$  in  $B$ , then  $I \otimes B + A \otimes J$  annihilates  $L_1 \otimes L_2$ . And  $A \otimes B / (I \otimes B + A \otimes J) \cong ((A \otimes B) / ((I \otimes B)(A \otimes J))) / (I \otimes B + A \otimes J) \cong (A/I) \otimes B / (A/I) \otimes J \cong (A/I) \otimes (B/J)$  so its one dimensional so it is a maximal ideal.

13. (a) Let  $X = \{(x, y) \in F^2 \mid xy = 0\}$ . Show that  $X$  is a closed, connected subset of  $F^2$  in the Zariski topology. What are its irreducible components?

(b) Show that the Zariski topology on  $F^2$  is *not* the same as the product topology on  $F \times F$  arising from the Zariski topology on each copy of  $F$ .

(a) Its closed by definition. Its irreducible components are  $V(x)$  and  $V(y)$ , since  $(xy) = (x) \cap (y)$  and  $(x)$  and  $(y)$  are prime ideals.

(b) In the Zariski topology on  $F$ , the proper closed sets are all finite. The same is therefore true for the product topology on  $F^2$ . But in the Zariski topology on  $F^2$  there are infinite proper closed sets...

14. Let  $(X, A)$  be an irreducible affine variety. It can be shown that if  $f \in A$  is any non-zero function, then its zero set  $V(f)$  has at least one irreducible component  $Y$  with  $\dim Y = \dim X - 1$  (a hypersurface!).

(a) Use this fact to prove that  $\dim X$  as defined in class is equal to the maximum number  $n$  such that there exists a strictly descending chain  $X \supset X_1 \supset \dots \supset X_n = \emptyset$  of closed irreducible subsets. (This is the usual definition of dimension of an irreducible noetherian topological space).

(b) Prove that  $\dim X$  as defined in class is equal to the maximal number  $n$  such that there exists a strictly increasing chain  $(0) = P_1 \subset \dots \subset P_n \subset A$  of prime ideals in  $A$ . (This is the usual definition of the Krull dimension of an integral domain).

(a) In class we proved that if  $Y$  is a proper closed irreducible subset of  $X$  and  $X$  is an irreducible affine variety, then  $\dim Y < \dim X$ . Hence if there is such a chain you get that  $\dim X \leq n$ .

Conversely, by the given fact, you can find a chain where each is codimension 1 in the previous subspace. So  $\dim X \geq n$ .

(b) This is exactly the same statement rephrased using algebra instead of geometry, using the Nullstellensatz and the fact that irreducible closed sets correspond to prime ideals.