Exercises on chapter 5, part III

8. In all the remaining questions, $F$ is an algebraically closed field and all algebras are commutative $F$-algebras.
(a) Prove that any prime ideal in the algebra $F[x_1,\ldots,x_n]$ is a radical ideal.
(b) For any ideal $I$ of $F[x_1,\ldots,x_n]$, prove that
$$\sqrt{I} = \bigcap_{I \subseteq J_{\text{max}}} J$$
where the intersection is over all maximal ideals $J$ containing $I$. (Hint: $\sqrt{I} = I(V(I)).$)
(c) For any affine algebra $A$ prove that its Jacobson radical $J(A)$ is $(0)$.

9. (a) Let $X$ be a topological space and let $S$ be a subset with the subspace topology. Prove that $S$ is an irreducible topological space if and only if its closure $\overline{S}$ in $X$ is.
(b) Let $f : X \rightarrow Y$ be a continuous map between topological spaces. Let $S$ be an irreducible subset of $X$ (i.e. $S$ is irreducible in the subspace topology). Prove that $f(S)$ is an irreducible subset of $Y$.
(c) Let $A$ and $B$ be affine algebras and $f : A \rightarrow B$ be an algebra homomorphism. Let $I$ be a prime ideal in $B$. Prove that $f^{-1}(I)$ is a prime ideal in $A$. What has this got to do with (b)?
(d) Let $I$ be a radical ideal in an affine algebra $A$. Prove that $I$ can be expressed as an intersection of finitely many prime ideals.

10. Let $A$ and $B$ be finitely generated algebras. (Commutative and over $F$, algebraically closed, of course.) I call a ring reduced if it has no non-zero nilpotent elements.
(a) If $A$ and $B$ are both reduced, prove that $A \otimes_F B$ is reduced too.
(b) If $A$ and $B$ are both integral domains, prove that $A \otimes_F B$ is an integral domain. (This is needed in class to show that the product of two irreducible affine varieties is irreducible.)
(c) Is $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ an integral domain?

11. Describe all irreducible Hausdorff topological spaces.

12. Let $A$ and $B$ be affine algebras. Prove that the maximal ideals of $A \otimes_F B$ are all of the form $A \otimes I + J \otimes B$ for $I$ a maximal ideal of $B$ and $J$ a maximal ideal of $A$. (This is needed in class to show that the product $X \times Y$ of two affine varieties is an affine variety with coordinate algebra $F[X] \otimes_F F[Y]$.) (Hint: it maybe easier to think in terms of irreducible modules.)

13. (a) Let $X = \{(x, y) \in F^2 \mid xy = 0\}$. Show that $X$ is a closed, connected subset of $F^2$ in the Zariski topology. What are its irreducible components?
(b) Show that the Zariski topology on $F^2$ is not the same as the product topology on $F \times F$ arising from the Zariski topology on each copy of $F$. 
14. Let \((X, A)\) be an irreducible affine variety. It can be shown that if \(f \in A\) is any non-zero function, then its zero set \(V(f)\) has at least one irreducible component \(Y\) with \(\dim Y = \dim X - 1\) (a hypersurface).

(a) Use this fact to prove that \(\dim X\) as defined in class is equal to the maximum number \(n\) such that there exists a strictly descending chain \(X \supset X_1 \supset \cdots \supset X_n = \emptyset\) of closed irreducible subsets. (This is the usual definition of dimension of an irreducible noetherian topological space).

(b) Prove that \(\dim X\) as defined in class is equal to the maximal number \(n\) such that there exists a strictly increasing chain \((0) = P_1 \subset \cdots \subset P_n \subset A\) of prime ideals in \(A\). (This is the usual definition of the Krull dimension of an integral domain).