Exercises on chapter 5, part III

8. *In all the remaining questions, F is an algebraically closed field and all algebras are commutative F-algebras.*

   (a) Prove that any prime ideal in the algebra $F[x_1, \ldots, x_n]$ is a radical ideal.

   (b) For any ideal I of $F[x_1, \ldots, x_n]$, prove that

   $$\sqrt{I} = \bigcap_{I \subseteq J \, max} J$$

   where the intersection is over all maximal ideals J containing I. (Hint: $\sqrt{I} = I(V(I))$.)

   (c) For any affine algebra A prove that its Jacobson radical $J(A)$ is (0).

9. (a) Let X be a topological space and let S be a subset with the subspace topology. Prove that S is an irreducible topological space if and only if its closure $\overline{S}$ in X is.

   (b) Let $f : X \to Y$ be a continuous map between topological spaces. Let S be an irreducible subset of X (i.e. S is irreducible in the subspace toplogy). Prove that $f(S)$ is an irreducible subset of Y.

   (c) Let A and B be affine algebras and $f : A \to B$ be an algebra homomorphism. Let I be a prime ideal in B. Prove that $f^{-1}(I)$ is a prime ideal in A. What has this got to do with (b)?

   (d) Let I be a radical ideal in an affine algebra A. Prove that I can be expressed as an intersection of finitely many prime ideals.

10. Let A and B be finitely generated algebras. (Commutative and over F, algebraically closed, of course.) I call a ring *reduced* if it has no non-zero nilpotent elements.

    (a) If A and B are both reduced, prove that $A \otimes_F B$ is reduced too.

    (b) If A and B are both integral domains, prove that $A \otimes_F B$ is an integral domain. (This is needed in class to show that the product of two irreducible affine varieties is irreducible.)

    (c) Is $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ an integral domain?

11. Describe all irreducible Hausdorff topological spaces.

12. Let A and B be affine algebras. Prove that the maximal ideals of $A \otimes_F B$ are all of the form $A \otimes I + J \otimes B$ for I a maximal ideal of B and J a maximal ideal of A. (This is needed in class to show that the product $X \times Y$ of two affine varieties is an affine variety with coordinate algebra $F[X] \otimes_F F[Y]$.) (Hint: it maybe easier to think in terms of irreducible modules.)

13. (a) Let $X = \{(x, y) \in F^2 \mid xy = 0\}$. Show that X is a closed, connected subset of F^2 in the Zariski topology. What are its irreducible components?

    (b) Show that the Zariski topology on F^2 is *not* the same as the product topology on $F \times F$ arising from the Zariski topology on each copy of F.

14. Let (X, A) be an irreducible affine variety. It can be shown that if $f \in A$ is any non-zero function, then its zero set $V(f)$ has at least one irreducible component Y with $\dim Y = \dim X - 1$ (a hypersurface!).

(a) Use this fact to prove that $\dim X$ as defined in class is equal to the maximum number n such that there exists a strictly descending chain $X \supset X_1 \supset \cdots \supset X_n = \varnothing$ of closed irreducible subsets. (This is the usual definition of dimension of an irreducible noetherian topological space).

(b) Prove that $\dim X$ as defined in class is equal to the maximal number n such that there exists a strictly increasing chain $(0) = P_1 \subset \cdots \subset P_n \subset A$ of prime ideals in A. (This is the usual definition of the Krull dimension of an integral domain).