Exercises on chapter 4

Always R-algebra means associative, unital R-algebra. (There are other sorts of R-algebra but we won't meet them in this course.)

1. Let A and B be algebras over a field F.

   (i) Explain how to make the vector space $A \otimes_F B$ into an F-algebra so that multiplication satisfies $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$ for all $a, a' \in A, b, b' \in B$. Did you think about well-definedness?

   (ii) Assuming that A and B are commutative, prove that $A \otimes B$ together with the maps $A \to A \otimes B, a \mapsto a \otimes 1_B$ and $B \to A \otimes B, b \mapsto 1_A \otimes B$ is the coproduct of A and B in the category of commutative F-algebras.

   (iii) Let $A = S(V)$ and $B = S(W)$ for vector spaces V and W. Prove that $S(V) \otimes S(W) \cong S(V \oplus W)$.

   (iv) Prove using (iii) that $F[x_1, \dots, x_n] \cong F[x] \otimes \cdots \otimes F[x]$ (n times).

   (i) Define a map $A \times B \times A \times B \to A \otimes B$ by $(a, b, a', b') \mapsto aa' \otimes bb'$. This is multilinear so induces a unique map $(A \otimes B) \otimes (A \otimes B) \to A \otimes B$, i.e. a bilinear multiplication on the vector space $A \otimes B$ such that $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

   (ii) Take a commutative algebra C and maps $f : A \to C, g : B \to C$. Define a map $A \times B \to C$ by $(a, b) \mapsto f(a)g(b)$. Its bilinear so induces a map $h : A \otimes B \to C$. Note that h is an algebra homomorphism by direct check on generators, using that C is commutative: $h((a \otimes b)(a' \otimes b')) = f(aa')g(bb') = f(a)f(a')g(b)g(b') = f(a)g(b)f(a')g(b')' = h(a \otimes b)h(a' \otimes b')$. And it is clearly the unique thing that makes the diagrams commute...

   (iii) Define a map $V \oplus W \to S(V) \otimes S(W)$ so that $v \in V \subset V \oplus W$ maps to $v \otimes 1$ and $w \in W$ maps to $1 \otimes w$. Since $S(V) \otimes S(W)$ is commutative, this induces by the universal property of symmetric algebra an algebra map $S(V \oplus W) \to S(V) \otimes S(W)$. It is surjective since its image contains generators for the latter algebra. To see that its injective its easiest to construct just a linear map $S(V) \otimes S(W) \to S(V \oplus W)$ which is the two-sided inverse of the map gotten so far. To do this start from the map $S(V) \times S(W) \to S(V \oplus W)$ mapping (f, g) to fg... here I'm thinking of $S(V)$ as sitting inside $S(V \oplus W)$ in obvious way etc...

   (iv) Let V be vector space on basis $x_1, \dots, x_n$. Then $F[x_1, \dots, x_n) = S(V)$. But $V \cong F \oplus \cdots \oplus F$ (n times) so $S(V)$ is isomorphic to $S(F) \otimes \cdots \otimes S(F)$ (n times), and of course $S(F) \cong F[x]$. Overall x_i maps to $1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ where x is in the ith slot.

2. Let V and W be finite dimenional vector spaces over a field F.

   (i) Compute the dimension of the exterior algebra $\bigwedge(V)$.

   (ii) Show that $\dim \bigwedge(V \oplus W) = \dim \bigwedge(V) \otimes \bigwedge(W)$.

   (iii) Is it true that $\bigwedge(V \oplus W) \cong \bigwedge(V) \otimes \bigwedge(W)$?

   (iv) Is it true that $T(V \oplus W) \cong T(V) \otimes T(W)$?

   (i) Its $2^{\dim V}$. If $e_1, \dots, e_n$ is a basis for V, the strict monomial basis of $\bigwedge(V)$ is indexed by all subsets of $\{1, \dots, n\}$, and there are 2^n such.

   (ii) $2^{m+n} = 2^m 2^n$.

(iii) No. Take V, W one dimensional. Then $\bigwedge(V)$ and $\bigwedge(W)$ are commutative. Hence so is their tensor product. But $\bigwedge(V \oplus W)$ is not commutative. (Unless that is the field has characteristic 2 when in fact it is true that these algebras are isomorphic!)

(iv) No. Same counterexample as (iii). One algebra is commutative, the other certainly is not.

3. Let G be a finite group and $\mathbb{C}G$ be its group algebra, i.e. the $\mathbb{C}$-vector space on basis G with bilinear multiplication induced by the multiplication in G. Prove that the dimension of the center $Z(\mathbb{C}G)$ of the algebra $\mathbb{C}G$ is equal to the number of conjugacy classes in the group G. (Hint: think about $\sum_{g \in C} g$ for C a conjugacy class in G.)

   A basis for the center is given by the class sums, one for each conjugacy class. Hence the dimension of the center is the number of conjugacy classes.

4. (i) Construct an isomorphism between the space of all symmetric bilinear forms on a finite dimenional vector space V and the space $S^2(V)^*$.

   (ii) Construct an isomorphism between the space of all alternating bilinear forms (i.e. $(v, v) = 0$ for all $v \in V$) and the space $\bigwedge^2(V)^*$.

   (i) Let b be a symmetric bilinear form on V. Define a map $V \times V \to F$ by $(v, w) \mapsto b(v, w)$. Its symmetric and bilinear, hence induces a unique linear map $S^2(V) \to F$, i.e. an element of $S^2(V)^*$. Now check its an isomorphism...

   (ii) Similar.

5. If F is a field of characteristic 0 and t is an indeterminate, then the ring $\mathrm{End}_F(F[t])$ contains the operators

$$x : f(t) \mapsto tf(t).$$

   and

$$y : f(t) \mapsto \frac{d}{dt} f(t)$$

   Let A be the subalgebra of $\mathrm{End}_F(F[t])$ generated by x and y. The is the *first Weyl algebra*.

   (i) Prove that $yx - xy = 1$.

   (ii) Prove that A has basis $\{x^i y^j \mid i, j \geq 0\}$. (It looks like the polynomial ring $F[x, y]$ but x and y don't quite commute...)

   (iii) Prove that $A \cong F\langle x, y\rangle/I$ where I is the two-sided ideal generated by the element $yx - xy - 1$.

   (iv) Prove that A is simple, i.e. it has no two-sided ideals other than (0) and A itself.

   (i) Check that $(yx - xy)(f(t)) = f(t)$ for any polynomial $f(t)$.

   (ii) Say $f \in A$ is of *degree* d if it is a linear combination of monomials in x and y of total degree $\leq d$. For example, $xyx + yx + x + yxy$ is of degree 3 (and of degree 4, 5, ...). Note any element of A is of degree n for some n. Now I show by induction on d that any element of A of degree d can be written as a linear combination of $\{x^i y^j \mid i, j \geq 0, i + j \leq d\}$. Well we just need to take a monomial $x^{i_1} y^{i_2} \cdots y^{i_n}$ of degree d which doesn't have all x's at the beginning. Using (i) you can commute yx to xy *modulo terms of degree* $(d - 1)$. By induction these lower terms can be expressed as a linear combination of the desired monomials. Done.

   So the $\{x^i y^j\}$ span A. We still need to show they are linearly independent. Say $\sum_{i,j \geq 0} a_{i,j} x^i y^j = 0$ but not all $a_{i,j}$ are zero. Pick j minimal such that $a_{i,j} \neq 0$ for some i. Apply to t^j. Note all y^k for $k > j$ kill t^j and y^j sends t^j to a non-zero scalar. So we just get that $\sum_{i \geq 0} a_{i,j} t^i = 0$. Since the t^i are linearly independent, this shows all these $a_{i,j} = 0$, a contradiction.

   (iii) Define a map $F\langle x, y\rangle \twoheadrightarrow A$ mapping x to x, y to y. By (i) this factors to a map $F\langle x, y\rangle/I \twoheadrightarrow A$. By (ii) the monomials $\{x^i y^j\}$ are independent in A, but they span $F\langle x, y\rangle/I$. Hence this is an isomorphism.

   (iv) Let $\partial\partial y : A \to A$ be the linear map $f \mapsto xf - fx$ for any $f \in A$. Clearly this leaves any two-sided ideal of A invariant. Note that $\partial\partial y x^i y^j = j x^i y^{j-1}$.

Now suppose I is any non-zero two-sided ideal. Take $0 \neq f = \sum a_{i,j} x^i y^j \in I$. Let j be maximal such that $a_{i,j} \neq 0$ for some i. Apply $\partial^j \partial y^j$ to deduce that some non-zero polynomial in x belongs to I too.

Now play the same game with roles of x and y switched, to see that you can differentiate that polynomial in x some more with respect to x to deduce that some non-zero contant lies in I. But that is a unit, hence $I = A$.

6. An R-module is called *noetherian* if it has the ascending chain condition on submodules, i.e. every chain $M_1 \leq M_2 \leq \cdots$ of submodules eventually stabilizes with $M_n = M_{n+1} = \cdots$ for some n. A ring R is *left noetherian* if ${}_RR$ is noetherian, i.e. it has ACC on left ideals. Similarly R is *right noetherian* if R_R is noetherian, i.e. it has ACC on right ideals. Prove that the ring of all matrices of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ for $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$ is left noetherian but not right noetherian.

   To prove that R is left noetherian, consider a left ideal $0 \neq I$. Suppose that $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ is a non-zero element of I. Multiply on the left by $e_{2,2}$ to deduce that I contains as a submodule some ideal J that is a submodule of the left ideal generated by $\begin{bmatrix} 0 & 0 \\ b & c \end{bmatrix}$ for all $b, c \in \mathbb{Q}$. That is a semisimple left ideal (it looks like two copies of $\mathbb{Q}$). Hence J has finite length. Then you can pass to I/J. If it is zero we're done. Otherwise, I/J looks like a copy of $\mathbb{Z}$ (only the $1,1$ entries of matrices play any role here), which is noetherian.

   To prove that R is not right noetherian, consider for each $i \geq 1$ the right ideal J_i generated by the matrix $\begin{bmatrix} \frac{1}{2^i} & 0 \\ 0 & 0 \end{bmatrix}$. We have that $J_1 < J_2 < J_3 < \ldots$ but each is stricly larger. The point is you can only multiply the $1,1$ entry by integers...

7. An R-module is called *artinian* if it has the descending chain condition on submodules, i.e. every chain $M_1 \geq M_2 \geq \cdots$ of submodules eventually stabilizes with $M_n = M_{n+1} = \cdots$ for some n. A ring R is *left artinian* if ${}_RR$ is artinian, i.e. it has DCC on left ideals. Similarly R is *right artinian* if R_R is artinian, i.e. it has DCC on right ideals. Prove that the ring of all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for $a \in \mathbb{Q}$ and $b, c \in \mathbb{R}$ is right artinian but not left artinian.

   Similar to 6.

8. For any ring R prove that $M_n(R)^{op} \cong M_n(R^{op})$.

   Map a matrix $A \in M_n(R)^{op}$ to the transpose matrix $A^T \in M_n(R^{op})$. (Of course the op's don't mean anything yet!). Now check this is a ring homomorphism: for matrices A, B, we have that

   $$A \cdot B = BA$$

   which has ij-entry $\sum_k b_{i,k} a_{k,j} = \sum_k a_{k,j} \cdot b_{k,i}$. Hence $(A \cdot B)^T$ has ij-entry $\sum_k a_{k,i} \cdot b_{k,j}$. On the other hand, $A^T B^T$ (usual matrix product but coefficients multiplied in $R^o p$) has ij-entry $a_{k,i} \cdot b_{k,j}$. Okay that was a little confusing...

9. If $r \in R$ is a nilpotent element of R (i.e. $r^m = 0$ for some $m > 0$), is it true that the left ideal I of R generated by r is nilpotent (i.e. $I^m = 0$ for some $m > 0$). Prove or give a counterexample.

   Take $R = M_2(F)$ and $r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This is a nilpotent element. But the left ideal it generates contains $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so it cannot possibly be a nilpotent ideal.

10. Compute the Jacobson radicals of the following rings.

    (i) $F[x]/(x^n)$ for $n > 0$ and F any field.

(ii) $\bigwedge(V)$ for V a finite dimensional vector space over a field F.

(iii) $S(V)$ for V a finite dimensional vector space over an infinite field F. (Hint: start by finding some one dimensional irreducible $S(V)$-modules.)

(i) There is just one irreducible module, namely the field F with x acting as 0. (Structure theory of modules over PIDs...) Its annihilator is the ideal (x). This must be the Jacobson radical.

(ii) It is finite dimensional, so artinian. So we can use the characterization of the Jacobson radical as the maximal nilpotent ideal. Clearly $\bigoplus_{i>0} \bigwedge^i(V)$ is a nilpotent ideal of codimension 1 in $\bigwedge(V)$. Therefore its got to be the Jacobson radical.

(iii) Note $S(V) \cong F[x_1, \dots, x_n]$ where $n = \dim V$. Given scalars $a_1, \dots, a_n$ we get a maximal ideal of $F[x_1, \dots, x_n]$, namely, $(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$. This is the annihilator of the one dimensional irreducible $F[x_1, \dots, x_n]$-module on which each x_i acts as multiplication by a_i. Now let $b_1, \dots, b_n$ be some more scalars with $b_i \neq a_i$. Then the intersection of $(x_1 - a_1, \dots, x_n - a_n)$ with $(x_1 - b_1, \dots, x_n - b_n)$ is $((x_1 - a_1)(x_1 - b_1), \dots, (x_n - a_n)(x_n - b_n))$. Keep going picking more and more distinct elements of the field. You get that the intersection of all these maximal ideals is an ideal generated by homogeneous polynomials of arbitrarily large degree. So it must be ZERO.

11. Let R be a left Artinian ring and M be a left R-module. Prove that there is a unique smallest submodule K of M (called the *radical* of M) such that M/K is a semisimple R-module.

Let J be the Jacobson radical of R. Let K be a submodule such that M/K is semisimple. Clearly then J annihilates M/K, i.e. $JM \subseteq K$. On the other hand, M/JM is annihilated by J so by a corollary from class it is a semisimple module. This shows that JM is the unique smallest submodule of M with semisimple quotient...

12 . Let A be an algebra over a field F.

(i) A *representation* of A means a pair (V, ρ) where V is a vector space and $\rho : A \to \mathrm{End}_F(V)$ is an algebra homomorphism. A morphism $f : (V, \rho) \to (W, \sigma)$ between two representations of A means a linear map $f : V \to W$ such that $f \circ \rho(a) = \sigma(a) \circ f$ for all $a \in A$. This defines the category $\mathrm{Rep}(A)$ of all representations of A. Prove that the category $\mathrm{Rep}(A)$ is isomorphic to the category A-mod.

(ii) A *matrix representation* of A means a ring homomorphism $\rho : A \to M_n(F)$ for some $n \geq 0$. Morphisms of matrix representations are defined in the same way as (i). This defines the category $\mathrm{Mat}(A)$ of matrix representations of A. Prove that the category $\mathrm{Mat}(A)$ is equivalent to the category of all finite dimensional A-modules. Could I replace the word equivalent with isomorphic here?

(i) Define a functor from $\mathrm{Rep}(A)$ to $A-mod$ by mapping (V, ρ) to the vector space V with A-module structure given by $av := \rho(a)(v)$. On morphisms you map a linear map $f : V \to W$ to the same function, it is automatically then an A-module homomorphism.

Define a functor from $A - mod$ to $\mathrm{Rep}(A)$ by mapping an A-module V to the pair (V, ρ) defined from $\rho(a)(v) := av$. Its the identity on morphisms.

Clearly these are mutually inverse isomorphisms of categories.

(ii) The subtlety is that its only an equivalence of categories not an isomorphism of categories. Let me first explain exactly what a morphism is in the category $Mat(A)$. Say $\rho : A \to M_n(F)$ and $\sigma : A \to M_m(F)$ are two matrix representations. A morphism $f : \rho \to \sigma$ means simply an $m \times n$ matrix M with the property that $M\rho(a) = \sigma(a)M$ for all $a \in A$. Compare with the definition in (i)... In matrix land linear maps are given by multiplying by rectangular matrices.

Now suppose we are given a matrix representation $\rho : A \to M_n(F)$ (the data is n and the function ρ...). Let $V = F^n$ be the vector space of column vectors. Define an A-module structure on V by $av := \rho(a)v$ (matrix multiplication). This defines a functor $F : Mat(A) \to A - fmod$ ($fmod$ means finite dimensional modules). At least, I've defined F on objects but its pretty clear what to do on morphisms too.

On the other hand there is a functor $G : A - fmod \to Mat(A)$. For each vector space V, pick once and for all a basis $v_1, \dots, v_n$ for V. Make sure for the canonical vector spaces F^n of column vectors that you chose the standard basis. This is horrible, of course. I'll always work with these fixed choices of bases from now on. Now take a finite dimensional A-module V, let $v_1, \dots, v_n$ be the basis we just chose for that vector space. Then define $G(V)$ to be the matrix representation $\rho : A \to M_n(F)$ defined by setting $\rho(a)$ to be the matrix representing the map $v \mapsto av$ in the chosen basis. For a morphism $f : V \to W$ let $G(f)$ be the matrix defining the linear map f in the chosen bases.

Now note that $G \circ F$ simply equal to the identity functor on $Mat(A)$ (because we chose the basis for F^n to be the canonical basis). On the other hand $F \circ G$ maps a vector space V on basis $v_1, \dots, v_n$ to the vector space F^n. There is an isomorphism $\eta_V : V \to F^n$ mapping v_i to the ith standard basis vector of F^n. This defines a natural transformation η from the identity functor on $A - fmod$ to $F \circ G$ which is clearly an isomorphism on each object, so $F \circ G \cong Id$.

13. The goal of this problem is to prove *Burnside's theorem*: If A is an algebra over an algebraically closed field F and V is a finite dimensional simple A-module, then the corresponding representation $\rho : A \to \mathrm{End}_F(V)$ is surjective. Note $\mathrm{End}_F(V)$ is isomorphic to the algebra $M_n(F)$ where $n = \dim V$...

(i) Prove that the algebra $\rho(A)$ is a semisimple algebra having just one simple module up to isomorphism, namely, V. (Hint: $\rho(A)$ is a subalgebra of $\mathrm{End}_F(V)$ – how does the latter decompose as left modules?)

(ii) Deduce from Wedderburn's theorem that $\rho(A)$ is isomorphic to $M_n(F)$ where $n = \dim V$. Hence $\rho(A) = \mathrm{End}_F(V)$ by dimensions.

(iii) Suppose that A is a subalgebra of the algebra $M_n(F)$ such that the natural module F^n of column vectors is irreducible as an A-module. What is A?

(i) $\rho(A)$ is a subalgebra of $\mathrm{End}_F(V)$. That is just the algebra of $n \times n$ matrices which as a left module over itself is a direct sum of n copies of V. Hence the left regular module of $\rho(A)$ is a submodule of a direct sum of n copies of the simple $\rho(A)$-module V, i.e. it is a submodule of a semisimple module so it is semisimple. Moreover, all simple $\rho(A)$-modules are submodules of its left regular module, hence submodules of a direct sum of n copies of V. So there is only one simple module up to isomorphism, namely, V.

(ii) So by the condition (ii) implies (iii) in our statement of Wedderburn, $\rho(A)$ is isomorphic to a matrix algebra of size $\dim V$ with entries in F. But then $\dim \rho(A) = \dim \mathrm{End}_F(V)$ so $\rho(A) = \mathrm{End}_F(V)$.

(iii) Apply the above to $V = F^n$. The representation $\rho : A \to \mathrm{End}_F(V)$ is onto by above, hence $\dim A \geq n^2 = \dim M_n(F)$. So since A is a subalgebra of $M_n(F)$ it must be all of $M_n(F)$ by dimension...

14. Suppose that $F \subset K$ are two fields and A is a finite dimensional F-algebra. Note then that $K \otimes_F A$ is a finite dimensional K-algebra and for any A-module M, $K \otimes_F M$ is naturally a $K \otimes_F A$-module.

(i) For any finite dimensional A-module M, show that

$$\mathrm{End}_{K \otimes_F A}(K \otimes_F M) \cong K \otimes_F \mathrm{End}_A(M).$$

(ii) A finite dimensional irreducible A-module M is called *absolutely irreducible* if the $K \otimes_F A$-module $K \otimes_F M$ is irreducible for every field extension K of F. If M is absolutely irreducible, prove that $\mathrm{End}_F(M) \cong F$ (Hint: take K to be algebraically closed).

(iii) If M is a finite dimensional irreducible A-module such that $\mathrm{End}_F(M) \cong F$, prove that the corresponding representation $\rho : A \to \mathrm{End}_F(M)$ is surjective. Hence show that M is absolutely irreducible.

(i) Define a map $K \times \mathrm{End}_A(M) \to \mathrm{End}_{K \otimes_F A}(K \otimes_F M)$ by sending (k, f) to the endomorphism $k \otimes f$, i.e. the tensor product of the maps $k : K \to K, c \mapsto kc$ and the map $f : M \to M$. It induces a map $K \otimes_F \mathrm{End}_A(M) \to \mathrm{End}_{K\otimes_F A}(K \otimes_F M)$, which is an algebra homomorphism. Note $(k \otimes f)(c \otimes m) = kc \otimes f(m)$...

To prove its an isomorphism, lets construct a map going the other way. Take an element of $\mathrm{End}_{K\otimes_F A}(K \otimes_F M)$. Its restriction to $M = 1 \otimes M \subset K \otimes_F M$ defines a map $f : M \to K \otimes_F M$. Pick a basis for K as an F-vector space, say $k_i (i \in I)$, and a basis $m_1, \ldots, m_n$ for M. Define a linear map $f_i : M \to M$ from $f(m) = \sum_{i\in I} k_i \otimes f_i(m)$. Note for fixed m that $f_i(m) \neq 0$ for only finitely many i (otherwise this doesn't make sense as a sum). In fact since f_i is linear and M is finite dimensional, only finitely many f_i are zero overall. So it makes sense to consider $\sum_{i\in I} k_i \otimes f_i$ in $K \otimes_F \mathrm{End}_F(M)$. This is the map the other way...

The two maps are inverse to one another. For instance, if we started with an element of $\mathrm{End}_{K\otimes_F A}(K \otimes_F M)$ as in the second paragraph, and that mapped to $\sum_{i\in I} k_i \otimes f_i$. Lets apply the map in the first paragraph to this. We get back a map which sends $1 \otimes m$ to $\sum_{i\in I} k_i \otimes f_i(m) = f(m)$ as required.

(ii) Take K to be algebraically closed. Then $K \otimes_F \mathrm{End}_F(M) \cong \mathrm{End}_K(K \otimes_F M)$ by (i). Since its absolutely irreducible the right hand side is K by Schur's lemma. This means that the left hand side is K too. So $\mathrm{End}_F(M)$ must have been just one dimensional as an F-vector space in the first place, i.e. $\mathrm{End}_F(M) \cong F$.

(iii) The first part is exactly the same proof as 13(iii) ... in that question the field was algebraically closed but all that was used in the proof was that $\mathrm{End}_F(M) = F$.

Now, M is absolutely irreducible as an A-module if and only if it is absolutely irreducible as a $\rho(A)$-module. But $\rho(A) = \mathrm{End}_F(M)$. In other words we are just looking at a matrix algebra $\rho(A) = M_n(F)$ and its simple module $M = F^n$ is the module of column vectors. But for this when you apply $K\otimes_F$? you get $M_n(K)$ and the module K^n, which is still irreducible.

15. Let C_n denote the cyclic group of order n and let FC_n denote its group algebra over a field F.

(i) Prove that $FC_n \cong F[x]/(x^n - 1)$.

(ii) How many isomorphism classes of irreducible $\mathbb{C}C_n$-modules are there? What are their dimensions? How is this consistent with Wedderburn's structure theorem for the semisimple algebra $\mathbb{C}C_n$?

(iii) How many isomorphism classes of irreducible $\mathbb{Q}C_n$-modules are there up to isomorphism? What are their dimensions? How is this consistent with Wedderburn's structure theorem for the semisimple algebra $\mathbb{Q}C_n$?

(iv) Which of the irreducible modules in (ii) and (iii) are absolutely irreducible?

(i) Define a map $F[x] \to FC_n$ mapping x to a cyclic generator x of C_n. Since $x^n = 1$, the kernel of this map contains $(x^n - 1)$, so it factors to induce a map $F[x]/(x^n - 1) \twoheadrightarrow FC_n$. But both are of dimension n as vector spaces, so this is an isomorphism.

(ii) By the structure theorem of modules over PID's, the irreducibles correspond to the irreducible polynomials in $F[x]$ that divide $x^n - 1$. Since over $\mathbb{C}$ $x^n - 1$ splits into distinct linear factors, the number of irreducible $\mathbb{C}C_n$-modules (up to isomorphism) is therefore n and each of these are 1 dimensional. It means that $\mathbb{C}C_n$ is isomorphic to $\mathbb{C} \times \cdots \times \mathbb{C}$ (n times), i.e. n 1×1 matrix algebras.

(iii) Recall that $x^n - 1 = \prod_{d|n} \Phi_d(x)$ where $\Phi_d(x)$ is the dth cyclotomic polynomial (which is irreducible over $\mathbb{Q}$). So by (i) the number of irreducible modules is the number of divisors d of n, and their dimensions over $\mathbb{Q}$ are given by $\phi(d)$, where ϕ is Euler's ϕ-function. Note $n = \sum_{d|n} \phi(d)$.

For $d|n$, let ω_d be a primitive dth root of 1 and let L_d be the corresponding irreducible of dimension $\phi(d)$. The endomorphism algebra $\mathrm{End}_{FC_n}(L_d)$ must be a division algebra over $\mathbb{Q}$. In fact it must be $\mathbb{Q}(\omega_d)$, which is of dimension ϕ_d as a $\mathbb{Q}$-vector space. (The endomorphism algebra of L_d is the same as the endomorphism algebra of the regular module of the algebra $\mathbb{Q}[x]/(\Phi_d(x))$ which is just the algebra itself, i.e. the field $\mathbb{Q}$ with a root of $\Phi_d(x)$ adjoint; recall the roots of $\Phi_d(x)$ are all powers of ω_d.) Then the Wedderburn decomposition is that $\mathbb{Q}C_d \cong \prod_{d|n} \mathbb{Q}(\omega_d)$, i.e. 1×1 matrices over division algebras...

(iv) All of them in (ii) are absolutely irreducible. In (iii) since the irreducibles over $\mathbb{C}$ are all one dimensional (by (ii)) the absolutely irreducible ones are just the one dimensional ones, i.e. the ones corresponding to divisors $d|n$ such that $\phi(d) = 1$. Remember $\phi(pq) = \phi(p)\phi(q)$ if p and q are relatively prime integers. Also $\phi(p^n) = p^n - p^{n-1}$ for p prime. So the only way it is one is if $d = 1$ or 2. Ah yes, ± 1 are the only roots of unity that belong to $\mathbb{Q}$.

16. Let $Q_3 = \{1, \bar{1}, i, \bar{i}, j, \bar{j}, k, \bar{k}\}$ be the quaternion group of order 8. (So $ij = k, ji = \bar{k} = \bar{1}k$ etc... I think you can guess the other multiplication rules given this!)

(i) Use Wedderburn's theorem to prove that $\mathbb{C}Q_3 \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

(ii) Now consider the real group algebra $\mathbb{R}Q_3$. Its semisimple by Mashchke's theorem, but $\mathbb{R}$ is not algebraically closed so some division algebras might arise in the Wedderburn decomposition. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternions (your favorite division algebra.) Prove that the linear map $\mathbb{R}Q_3 \to \mathbb{H}$ defined on the basis vectors by "replacing the bars with minuses" ($i \mapsto i, \bar{i} \mapsto -i$, etc...) defines a surjective algebra homomorphism to the quaternions $\mathbb{H}$ (the 4-dimensional $\mathbb{R}$-algebra on basis $1, i, j, k$).

(iii) Find four non-isomorphic irreducible $\mathbb{R}Q_3$ modules each of $\mathbb{R}$-dimension 1.

(iv) Combining your answers, prove that $\mathbb{R}Q_3 \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}$ as an $\mathbb{R}$-algebra.

(v) Deduce from the uniqueness in Wedderburn's theorem that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$ as $\mathbb{C}$-algebras.

(vi) Can you think of another $\mathbb{R}$-algebra A not isomorphic to $\mathbb{H}$ as an $\mathbb{R}$-algebra such that $\mathbb{C} \otimes_{\mathbb{R}} A \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ as $\mathbb{C}$-algebras?

(i) Well its order 8, there are 5 conjugacy classes, and the only way of writing 8 as a sum of 5 squares is $1 + 1 + 1 + 1 + 4$.

(ii) It is.

(iii) Note $Q_3 \twoheadrightarrow V_4$, there are four irreducible V_4-modules. Their lifts give us what we're after, on all of them $\bar{1}$ acts as 1. (a) i acts as 1, j acts as 1; (b) i acts as -1, j acts as 1; (c) i acts as -1, j acts as -1; (d) i acts as 1, j acts as -1. All other things you can work out from this...

(iv) Now $\mathbb{R}Q_3$ is still semisimple, but the field is not algebraically closed so we cannot count irreducibles simply by counting conjugacy classes. But in (iii) we found four one dimensional representations, their endomorphism algebras must certainly all be $\mathbb{R}$. In (ii) we found a map $\mathbb{R}Q_3 \twoheadrightarrow \mathbb{H}$. The simple $\mathbb{H}$-module $\mathbb{H}$ therefore lifts to an irreducible $\mathbb{R}Q_3$-module. Its endomorphism algebra must be the division algebra $\mathbb{H}$ again. So using the Wedderburn theorem and thinking about dimension, we must have that $\mathbb{R}Q_3 \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{H}$.

(v) Well $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}Q_3 \cong \mathbb{C}Q_3$. Hence by (i) its isomorphic to $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$. But by (iv) its isomorphic to $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$. Hence by uniqueness in Wedderburn, $M_2(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$.

(vi) $M_2(\mathbb{R})$.

17. (i) Suppose that $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ according to Wedderburn's theorem. Explain how you can use the character table of G to find explicit formulae for mutually orthogonal central idempotents $e_1, \dots, e_r$ in $\mathbb{C}G$ summing to 1 such that $\mathbb{C}Ge_i \cong M_{n_i}(\mathbb{C})$.

(ii) Compute the character table of the cyclic group C_n.

(iii) Write down explicit formulae for mutually orthogonal central idempotents $e_1, \dots, e_n \in C_n$ such that

$$\mathbb{C}C_n \cong \mathbb{C}C_n e_1 \oplus \cdots \oplus \mathbb{C}C_n e_n$$

and each $\mathbb{C}C_n e_i \cong \mathbb{C}$.

(iv) What has your answer to (iii) got to do with the Chinese remainder theorem?

(i) $e_i = \sum_{g \in G} \frac{n_i \chi_i(g^{-1})}{|G|} g$. We proved it in class.

(ii) Let ω be a primitive nth root of 1. Ordering the columns $1, x, x^2, \dots, x^{n-1}$, the ith row of the character table is $1, \omega^i, \omega^{2i}, \dots, \omega^{2(n-1)}$.

(iii) $e_i = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{-ij} x^j$.

(iv) The Chinese Remainder Theorem says that $\mathbb{C}[x]/(x^n-1)$ is isomorhic to $\bigoplus_{i=0}^{n-1}\mathbb{C}[x]/(x-\omega^i)$. Since $\mathbb{C}C_n \cong \mathbb{C}[x]/(x^n-1)$ this is exactly what we proved in (iii).

18. (i) Prove that if z is a complex root of unity then $z^{-1}=\bar{z}$.

(ii) Prove that if G is a finite group and M is a finite dimensional $\mathbb{C}G$-module with character χ_M, then $\chi_M(g^{-1})=\overline{\chi_M(g)}$. (Hint: Every eigenvalue of $\rho(g)$ is a root of unity...)

(iii) If M is a finite dimensional $\mathbb{C}G$-module, its *dual* M^* is the dual vector space with action of $g\in G$ defined by $(gf)(v)=f(g^{-1}v)$ for $f\in M^*, v\in M$. Prove that M^* is irreducible if and only if M is irreducible.

(iv) Prove that the character χ_{M^*} of M^* is related to the character χ_M of M by $\chi_{M^*}(g)=\overline{\chi_M(g)}$. Deduce that $M\cong M^*$ if and only if $\chi_M(g)$ is a real number for every $g\in G$.

(v) Prove that $g\in G$ is conjugate to g^{-1} if and only if $\chi(g)$ is a real number for every character χ.

(i) $(e^{i\theta})^{-1}=e^{-i\theta}=\overline{e^{i\theta}}$.

(ii) Say the eigenvalues of $\rho(g)$ are $z_1,\ldots,z_n$. The eigenvalues of $\rho(g^{-1})$ are $z_1^{-1},\ldots,z_n^{-1}$. Each is a root of unity, so $z_i^{-1}=\overline{z_i}$. So $\chi(g)=z_1+\cdots+z_n$ and $\chi(g^{-1})=\overline{z_1}+\cdots+\overline{z_n}$.

(iii) If N is a non-zero proper submodule of M, then its annihilator in M^* is a non-zero proper submodule of M^*. So if M is reducible so is M^*. Hence if M^* is reducible so is $M^{**}\cong M$.

(iv) Let $v_1,\ldots,v_n$ be a basis for M, let $\rho: G\to GL_n(\mathbb{C})$ be the corresponding matrix representation. Take the dual basis $f_1,\ldots,f_n$ for M^*. The coresponding matrix representation maps g to $\rho(g^{-1})^T$. So $\chi_{M^*}(g)=\chi_M(g^{-1})=\overline{\chi_M(g)}$.

Hence $M\cong M^*$ if and only if $\chi_M=\chi_{M^*}$ if and only if $\chi_M(g)$ is invariant under complex conjugation for all $g\in G$.

(v) The irreducible characters form a basis for $C(G)$, as do the indicator functions of the conjugacy classes of G. Hence g and h lie in the same conjugacy class of G if and only if $\chi_i(g)=\chi_i(h)$ for all $i=1,\ldots,r$. Hence g and g^{-1} lie in the same conjugacy class of G if and only if $\chi(g)=\chi(g^{-1})=\overline{\chi(g)}$ for all characters χ.

19. If $g\in G$ show that $|C_G(g)|=\sum_{i=1}^r|\chi_i(g)|^2$. Conclude that the character table gives $|C_G(g)|$ for each $g\in G$.

By the column orthogonality relation, $\sum_{i=1}^r|\chi_i(g)|^2=|G|/c$ where c is the size of the conjugacy class of g. But $c=|C_G(g)|$.

20. Is the character table

`http://www.chemsoc.org/exemplarchem/entries/2004/ hull_booth/Special_groups/HighSym-Ih.htm`

(Printout attached) really a character table?

No. The rows are not linearly independent.

21. Compute the character table of the alternating group A_4.

See p.630 in Rotman.

22. Compute the character table of the dihedral group D_5.

Let $D_5=\{1,x,x^2,x^3,x^4,y,xy,x^2y,x^3y,x^4y\}$ with relation $yxy=x^4$. The conjugacy classes are $\{1\},\{x,x^4\},\{x^2,x^3\}$ and $\{y\}$. We get

C	1	x	x^2	y
$\lvert C\rvert$	1	2	2	5
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	a	b	0
χ_4	2	$-1-a$	$-1-b$	0

Now there's a map $D_4 \twoheadrightarrow C_2$ factoring out the normal subgroup of rotations. You get a non-trivial degree one character by lifting through this. Now the sums of the square of the degrees is 10, so the remaining two characters are of degree 2. Suppose $\chi_3(y) \neq 0$. Then $\chi_4 = \chi_3\chi_2$ and by the column orthogonality relations you rapidly get a contradiction. Hence $\chi_3(y) = \chi_4(y) = 0$. Now the remaining entries must something, call them a and b. Then use the orthogonality relations to get some equations, like $b = -a - 1$ and $a^2 + a - 1 = 0$. Solve this to get $a = \frac{1+\sqrt{5}}{2}, b = \frac{1-\sqrt{5}}{2}$...

23. Here are some partially completed character tables of some finite groups. I've also listed the orders of some of the conjugacy classes... Use what you know about character tables to fill in the remaining entries.

(i)

C	C_1	C_2	C_3	C_4	C_5
$\lvert C\rvert$	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2					
χ_3	3	-1	0	$\varepsilon^2+\varepsilon^3+1$	$\varepsilon+\varepsilon^4+1$
χ_4		0	1	-1	-1
χ_5				0	

(Here $\varepsilon = e^{2\pi i/5}$.)

(ii)

C	C_1	C_2	C_3	C_4	C_5	C_6
$\lvert C\rvert$						
χ_1	1	1	1	1	1	
χ_2	1	-1	-1	1	1	
χ_3	1	1	-1	-1	1	
χ_4	1	-1	1	-1	1	
χ_5	2	0	0	1	-1	
χ_6	2	0	0	-1	-1	

(iii)

C	C_1	C_2	C_3	C_4	C_5
$\lvert C\rvert$	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	
χ_4	1	1	-1	-1	
χ_5					

(iv)

C	C_1	C_2	C_3	C_4	C_5
$\lvert C\rvert$	1	6	8	6	3
χ_1	1	1	1	1	1
χ_2		-1		-1	1
χ_3	2			0	
χ_4	3	1			-1
χ_5	3			1	

(i)

C	C_1	C_2	C_3	C_4	C_5
$\lvert C\rvert$	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\varepsilon+\varepsilon^4+1$	$\varepsilon^2+\varepsilon^3+1$
χ_3	3	-1	0	$\varepsilon^2+\varepsilon^3+1$	$\varepsilon+\varepsilon^4+1$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

(ii)

C	C_1	C_2	C_3	C_4	C_5	C_6
$\|C\|$	1	3	3	2	2	1
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	1
χ_3	1	1	-1	-1	1	-1
χ_4	1	-1	1	-1	1	-1
χ_5	2	0	0	1	-1	-2
χ_6	2	0	0	-1	-1	2

(iii)

C	C_1	C_2	C_3	C_4	C_5
$\|C\|$	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	
χ_4	1	1	-1	-1	
χ_5	2	-2	0	0	0

(iv)

C	C_1	C_2	C_3	C_4	C_5
$\|C\|$	1	6	8	6	3
χ_1	1	1	1	1	1
χ_2	11	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

24. Recall from class how you use the character table of G to find the lattice of normal subgroups of G and their orders. Hence work out which of the groups whose character tables you computed in questions 21–23 are simple groups.

    Just 23(i).

25. (i) Explain how to use the character table of G to find $Z(G)$.

    (ii) Explain how to use the character table of G to determine whether G is nilpotent.

    (iii) Use the character tables to determine which of the groups whose character table you computed in 21–23 are nilpotent groups.

    (i) Note $Z(G)$ is the union of the conjugacy classes of size 1. So its all g such that $\sum |\chi_i(g)|^2 = |G|$, i.e. since $|\chi_i(g)| < \chi_i(1)$, its all g such that $|\chi_i(g)| = \chi_i(1)$ for all i.

    (ii) A finite group is nilpotent if and only if it has a normal Sylow p-group for each prime p. You can work out the order of G from the character table, and the lattice of normal subgroups together with their orders. So you can check this condition.

    (iii) I don't think any of them are.

26. Compute the character table of the symmetric group S_6.

    You start frm the permutation characters on singletons, pairs and triplets. It takes a while but you should be able to get

C	(1)	(12)	(123)	(12)(34)	(1234)	(12)(345)	(12345)	(123)(456)	(12)(3456)	(12)(34)(56)	(123456)
$\|C\|$	1	15	40	45	90	120	144	40	90	15	120
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1	1	1	-1	-1
χ_3	5	3	2	1	1	0	0	-1	-1	-1	-1
χ_4	5	-3	2	1	-1	0	0	-1	-1	1	1
χ_5	9	3	0	1	-1	0	-1	0	1	3	0
χ_6	9	-3	0	1	1	0	-1	0	1	-3	0
χ_7	5	1	-1	1	-1	1	0	2	-1	-3	0
$\chi_8$5	-1	-1	1	1	-1	0	2	-1	3	0	
χ_9	10	-2	1	-2	0	1	0	1	0	2	-1
χ_{10}	10	2	1	-2	0	-1	0	1	0	-2	1
χ_{11}	16	0	-2	0	0	0	1	-2	0	0	0

27. Let G be a finite group, $L_1, \dots, L_r$ be a set of representatives of the isomorphism classes of irreducible $\mathbb{C}G$-modules, $\chi_1, \dots, \chi_r$ be the corresponding irreducible characters.

(i) Prove that $\dim \operatorname{Hom}_{\mathbb{C}G}(L_i, L_j) = (\chi_i, \chi_j)$.

(ii) For any finite dimensional $\mathbb{C}G$-modules V and W, prove that $\dim \operatorname{Hom}_{\mathbb{C}G}(V, W) = (\chi_V, \chi_W)$.

(i) Well $(\chi_i, \chi_j) = \delta_{i,j}$ we know. Also $\operatorname{Hom}_{\mathbb{C}G}(L_i, L_i)$ is one dimensional by Schur's lemma. Finally for $i \neq j$ $\operatorname{Hom}_{\mathbb{C}G}(L_i, L_j) = 0$ since otherwise a non-zero map $L_i \to L_j$ would be an isomorphism...

(ii) Let $V = \bigoplus_{i=1}^r L_i^{\oplus a_i}$ and $W = \bigoplus_{i=1}^r L_i^{\oplus b_i}$ be decompositions of V and W as direct sums of irreducibles. Then $\chi_V = \sum a_i \chi_i$ and $\chi_W = \sum b_i \chi_i$, hence $(\chi_V, \chi_W) = \sum a_i b_i$. Similar calculation using one gives the same for $\dim \operatorname{Hom}_{\mathbb{C}G}(V, W)$.

28. Let H be a subgroup of a finite group G. Given any $\mathbb{C}H$-module V, let $\operatorname{ind}_H^G V$ denote the $\mathbb{C}G$-module $\mathbb{C}G \otimes_{\mathbb{C}H} V$. Given any $\mathbb{C}G$-module W, let $\operatorname{res}_H^G W$ denote the $\mathbb{C}H$-module obtained from W by restricting the action of G to the subgroup of H.

(i) Prove that

$$\dim \operatorname{Hom}_{\mathbb{C}G}(\operatorname{ind}_H^G V, W) = \dim \operatorname{Hom}_{\mathbb{C}H}(V, \operatorname{res}_H^G W).$$

(ii) Let $t_1, \dots, t_k$ be a set of G/H-coset representatives. Let $v_1, \dots, v_n$ be a basis for V. Prove that $\{t_i \otimes v_j \mid i = 1, \dots, k, j = 1, \dots, n\}$ is a basis for $\operatorname{ind}_H^G V$. How does $g \in G$ act on the basis element $t_i \otimes v_j$?

(iii) Let $\chi \in C(H)$ be the character of the $\mathbb{C}H$-module V. Define $\dot{\chi}(g) = \chi(g)$ if $g \in H$ or 0 if $g \in G - H$. Also let $\operatorname{ind}_H^G \chi \in C(G)$ be the character of the induced module $\operatorname{ind}_H^G V$. Using your answer to (ii), prove that

$$(\operatorname{ind}_H^G \chi)(g) = \sum_{i=1}^k \dot{\chi}(t_i^{-1} g t_i).$$

(iv) Let $\psi \in C(G)$ be the character of the $\mathbb{C}G$-module W, and let $\operatorname{res}_H^G \psi \in C(H)$ be its restriction to H. Use the result from problem 27 to prove *Frobenius reciprocity*:

$$(\operatorname{ind}_H^G \chi, \psi)_G = (\chi, \operatorname{res}_H^G \psi)_H$$

where $(.,.)_G$ and $(.,.)_H$ are the usual inner products on $C(G)$ and $C(H)$ respectively.

(i) Well by adjointness of tensor and hom, we have at once that

$$\operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{C}H} V, W) \cong \operatorname{Hom}_{\mathbb{C}H}(V, \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G, W)).$$

But also

$$\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, W) \cong \mathrm{res}_H^G W$$

as a $\mathbb{C}H$-module, the isomorphism being evaluation at 1. This does the job.

(ii) $\mathbb{C}G$ is a free right $\mathbb{C}H$-module on basis $t_1, \dots, t_r$. Hence $\mathbb{C}G \otimes_{\mathbb{C}H} V$ has the given basis... its just tensoring with a free module which is easy to understand like vector spaces.

To work out how g acts on $t_i \otimes v_j$, you have to write $gt_i = t_k h$ for some k and $h \in H$. Then $g(t_i \otimes v_j) = t_k \otimes hv_j$.

(iii) To compute $(\mathrm{ind}_H^G \chi)(g)$ we have to compute the trace of the map $t_i \otimes v_j \mapsto t_k \otimes hv_j$, where $gt_i = t_k h$, working with the basis from (ii). There's no contribution to the trace unless $k = i$, i.e. $t_i^{-1} g t_i \in h$. In that case, the contribution to the trace is the same as the trace of $t_i^{-1} g t_i$ acting on V. So you get exactly

$$\sum_{i=1}^{k} \dot{\chi}(t_i^{-1} g t_i)$$

as claimed.

(iv) By problem 27,

$$(\mathrm{ind}_H^G \chi, \psi)_G = \dim \mathrm{Hom}_{\mathbb{C}G}(\mathrm{ind}_H^G V, W).$$

Also

$$(\chi, \mathrm{res}_H^G \psi)_H = \dim \mathrm{Hom}_{\mathbb{C}H}(V, \mathrm{res}_H^G W).$$

By part (i) these two dimensions are the same.

29. Let H be a subgroup of G. Let χ be the trivial character of H. Let $\mathrm{ind}_H^G \chi$ be the induced character of G like in the previous question. Prove that $\mathrm{ind}_H^G \chi$ is the permutation character arising from the usual action of G by left multiplication on the set G/H of cosets of H in G. Thus induced characters generalize permutation characters...

Let $t_1, \dots, t_k$ be a set of G/H-coset representatives. So $G/H = \{t_1 H, \dots, t_k H\}$. The permutation character ψ of G on G/H satisfies $\psi(g) = \#\{i = 1, \dots, k | gt_i H = t_i H\} = \#\{i = 1, \dots, k | t_i^{-1} g t_i \in H\}$. Comparing with the formula from 28(iii) this is exactly $(\mathrm{ind}_H^G \chi)(g)$.