Exercises on chapter 4

Always R-algebra means associative, unital R-algebra. (There are other sorts of R-algebra but we won't meet them in this course.)

1. Let A and B be algebras over a field F.

   (i) Explain how to make the vector space $A \otimes_F B$ into an F-algebra so that multiplication satisfies $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$ for all $a, a' \in A, b, b' \in B$. Did you think about well-definedness?

   (ii) Assuming that A and B are commutative, prove that $A \otimes B$ together with the maps $A \to A \otimes B, a \mapsto a \otimes 1_B$ and $B \to A \otimes B, b \mapsto 1_A \otimes B$ is the coproduct of A and B in the category of commutative F-algebras.

   (iii) Let $A = S(V)$ and $B = S(W)$ for vector spaces V and W. Prove that $S(V) \otimes S(W) \cong S(V \oplus W)$.

   (iv) Prove using (iii) that $F[x_1, \dots, x_n] \cong F[x] \otimes \cdots \otimes F[x]$ (n times).

2. Let V and W be finite dimenional vector spaces over a field F.

   (i) Compute the dimension of the exterior algebra $\bigwedge(V)$.

   (ii) Show that $\dim \bigwedge(V \oplus W) = \dim \bigwedge(V) \otimes \bigwedge(W)$.

   (iii) Is it true that $\bigwedge(V \oplus W) \cong \bigwedge(V) \otimes \bigwedge(W)$?

   (iv) Is it true that $T(V \oplus W) \cong T(V) \otimes T(W)$?

3. Let G be a finite group and $\mathbb{C}G$ be its group algebra, i.e. the $\mathbb{C}$-vector space on basis G with bilinear multiplication induced by the multiplication in G. Prove that the dimension of the center $Z(\mathbb{C}G)$ of the algebra $\mathbb{C}G$ is equal to the number of conjugacy classes in the group G. (Hint: think about $\sum_{g \in C} g$ for C a conjugacy class in G.)

4. (i) Construct an isomorphism between the space of all symmetric bilinear forms on a finite dimenional vector space V and the space $S^2(V)^*$.

   (ii) Construct an isomorphism between the space of all alternating bilinear forms (i.e. $(v, v) = 0$ for all $v \in V$) and the space $\bigwedge^2(V)^*$.

5. If F is a field of characteristic 0 and t is an indeterminate, then the ring $\mathrm{End}_F(F[t])$ contains the operators

   $$x : f(t) \mapsto tf(t).$$

   and

   $$y : f(t) \mapsto \frac{d}{dt} f(t)$$

   Let A be the subalgebra of $\mathrm{End}_F(F[t])$ generated by x and y. The is the *first Weyl algebra.*

   (i) Prove that $yx - xy = 1$.

   (ii) Prove that A has basis $\{x^i y^j \mid i, j \geq 0\}$. (It looks like the polynomial ring $F[x, y]$ but x and y don't quite commute...)

(iii) Prove that $A \cong F\langle x, y\rangle/I$ where I is the two-sided ideal generated by the element $yx - xy - 1$.

(iv) Prove that A is simple, i.e. it has no two-sided ideals other than (0) and A itself.

6. An R-module is called *noetherian* if it has the ascending chain condition on submodules, i.e. every chain $M_1 \leq M_2 \leq \cdots$ of submodules eventually stabilizes with $M_n = M_{n+1} = \cdots$ for some n. A ring R is *left noetherian* if ${}_R R$ is noetherian, i.e. it has ACC on left ideals. Similarly R is *right noetherian* if R_R is noetherian, i.e. it has ACC on right ideals. Prove that the ring of all matrices of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ for $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$ is left noetherian but not right noetherian.

7. An R-module is called *artinian* if it has the descending chain condition on submodules, i.e. every chain $M_1 \geq M_2 \geq \cdots$ of submodules eventually stabilizes with $M_n = M_{n+1} = \cdots$ for some n. A ring R is *left artinian* if ${}_R R$ is artinian, i.e. it has DCC on left ideals. Similarly R is *right artinian* if R_R is artinian, i.e. it has DCC on right ideals. Prove that the ring of all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for $a \in \mathbb{Q}$ and $b, c \in \mathbb{R}$ is right artinian but not left artinian.

8. For any ring R prove that $M_n(R)^{op} \cong M_n(R^{op})$.

9. If $r \in R$ is a nilpotent element of R (i.e. $r^m = 0$ for some $m > 0$), is it true that the left ideal I of R generated by r is nilpotent (i.e. $I^m = 0$ for some $m > 0$). Prove or give a counterexample.

10. Compute the Jacobson radicals of the following rings.

(i) $F[x]/(x^n)$ for $n > 0$ and F any field.

(ii) $\bigwedge(V)$ for V a finite dimensional vector space over a field F.

(iii) $S(V)$ for V a finite dimensional vector space over an infinite field F. (Hint: start by finding some one dimensional irreducible $S(V)$-modules.)

11. Let R be a left Artinian ring and M be a left R-module. Prove that there is a unique smallest submodule K of M (called the *radical* of M) such that M/K is a semisimple R-module.

12 . Let A be an algebra over a field F.

(i) A *representation* of A means a pair (V, ρ) where V is a vector space and $\rho : A \to \mathrm{End}_F(V)$ is an algebra homomorphism. A morphism $f : (V, \rho) \to (W, \sigma)$ between two representations of A means a linear map $f : V \to W$ such that $f \circ \rho(a) = \sigma(a) \circ f$ for all $a \in A$. This defines the category $\mathrm{Rep}(A)$ of all representations of A. Prove that the category $\mathrm{Rep}(A)$ is isomorphic to the category A-mod.

(ii) A *matrix representation* of A means a ring homomorphism $\rho : A \to M_n(F)$ for some $n \geq 0$. Morphisms of matrix representations are defined in the same way as (i). This defines the category $\mathrm{Mat}(A)$ of matrix representations of A. Prove that the category $\mathrm{Mat}(A)$ is equivalent to the category of all finite dimensional A-modules. Could I replace the word equivalent with isomorphic here?

13. The goal of this problem is to prove *Burnside's theorem*: If A is an algebra over an algebraically closed field F and V is a finite dimensional simple A-module, then the corresponding representation $\rho : A \to \mathrm{End}_F(V)$ is surjective. Note $\mathrm{End}_F(V)$ is isomorphic to the algebra $M_n(F)$ where $n = \dim V$...

(i) Prove that the algebra $\rho(A)$ is a semisimple algebra having just one simple module up to isomorphism, namely, V. (Hint: $\rho(A)$ is a subalgebra of $\mathrm{End}_F(V)$ – how does the latter decompose as left modules?)

(ii) Deduce from Wedderburn's theorem that $\rho(A)$ is isomorphic to $M_n(F)$ where $n = \dim V$. Hence $\rho(A) = \mathrm{End}_F(V)$ by dimensions.

(iii) Suppose that A is a subalgebra of the algebra $M_n(F)$ such that the natural module F^n of column vectors is irreducible as an A-module. What is A?

14. Suppose that $F \subset K$ are two fields and A is a finite dimensional F-algebra. Note then that $K \otimes_F A$ is a finite dimensional K-algebra and for any A-module M, $K \otimes_F M$ is naturally a $K \otimes_F A$-module.

(i) For any finite dimensional A-module M, show that

$$\mathrm{End}_{K\otimes_F A}(K \otimes_F M) \cong K \otimes_F \mathrm{End}_A(M).$$

(ii) A finite dimensional irreducible A-module M is called *absolutely irreducible* if the $K \otimes_F A$-module $K \otimes_F M$ is irreducible for every field extension K of F. If M is absolutely irreducible, prove that $\mathrm{End}_F(M) \cong F$ (Hint: take K to be algebraically closed).

(iii) If M is a finite dimensional irreducible A-module such that $\mathrm{End}_F(M) \cong F$, prove that the corresponding representation $\rho : A \to \mathrm{End}_F(M)$ is surjective. Hence show that M is absolutely irreducible.

15. Let C_n denote the cyclic group of order n and let FC_n denote its group algebra over a field F.

(i) Prove that $FC_n \cong F[x]/(x^n - 1)$.

(ii) How many isomorphism classes of irreducible $\mathbb{C}C_n$-modules are there? What are their dimensions? How is this consistent with Wedderburn's structure theorem for the semisimple algebra $\mathbb{C}C_n$?

(iii) How many isomorphism classes of irreducible $\mathbb{Q}C_n$-modules are there up to isomorphism? What are their dimensions? How is this consistent with Wedderburn's structure theorem for the semisimple algebra $\mathbb{Q}C_n$?

(iv) Which of the irreducible modules in (ii) and (iii) are absolutely irreducible?

16. Let $Q_3 = \{1, \bar{1}, i, \bar{i}, j, \bar{j}, k, \bar{k}\}$ be the quaternion group of order 8. (So $ij = k, ji = \bar{k} = \bar{1}k$ etc... I think you can guess the other multiplication rules given this!)

(i) Use Wedderburn's theorem to prove that $\mathbb{C}Q_3 \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

(ii) Now consider the real group algebra $\mathbb{R}Q_3$. Its semisimple by Mashchke's theorem, but $\mathbb{R}$ is not algebraically closed so some division algebras might arise in the Wedderburn decomposition. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternions (your favorite division algebra.) Prove that the linear map $\mathbb{R}Q_3 \to \mathbb{H}$ defined on the basis vectors by "replacing the bars with minuses" ($i \mapsto i, \bar{i} \mapsto -i$, etc...) defines a surjective algebra homomorphism to the quaternions $\mathbb{H}$ (the 4-dimensional $\mathbb{R}$-algebra on basis $1, i, j, k$).

(iii) Find four non-isomorphic irreducible $\mathbb{R}Q_3$ modules each of $\mathbb{R}$-dimension 1.

(iv) Combining your answers, prove that $\mathbb{R}Q_3 \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}$ as an $\mathbb{R}$-algebra.

(v) Deduce from the uniqueness in Wedderburn's theorem that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$ as $\mathbb{C}$-algebras.

(vi) Can you think of another $\mathbb{R}$-algebra A not isomorphic to $\mathbb{H}$ as an $\mathbb{R}$-algebra such that $\mathbb{C} \otimes_{\mathbb{R}} A \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ as $\mathbb{C}$-algebras?

17. (i) Suppose that $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ according to Wedderburn's theorem. Explain how you can use the character table of G to find explicit formulae for mutually orthogonal central idempotents $e_1, \ldots, e_r$ in $\mathbb{C}G$ summing to 1 such that $\mathbb{C}Ge_i \cong M_{n_i}(\mathbb{C})$.

(ii) Compute the character table of the cyclic group C_n.

(iii) Write down explicit formulae for mutually orthogonal central idempotents $e_1, \dots, e_n \in C_n$ such that

$$\mathbb{C}C_n \cong \mathbb{C}C_n e_1 \oplus \cdots \oplus \mathbb{C}C_n e_n$$

and each $\mathbb{C}C_n e_i \cong \mathbb{C}$.

(iv) What has your answer to (iii) got to do with the Chinese remainder theorem?

18. (i) Prove that if z is a complex root of unity then $z^{-1} = \bar{z}$.

    (ii) Prove that if G is a finite group and M is a finite dimensional $\mathbb{C}G$-module with character χ_M, then $\chi_M(g^{-1}) = \overline{\chi_M(g)}$. (Hint: Every eigenvalue of $\rho(g)$ is a root of unity...)

    (iii) If M is a finite dimensional $\mathbb{C}G$-module, its *dual* M^* is the dual vector space with action of $g \in G$ defined by $(gf)(v) = f(g^{-1}v)$ for $f \in M^*, v \in M$. Prove that M^* is irreducible if and only if M is irreducible.

    (iv) Prove that the character χ_{M^*} of M^* is related to the character χ_M of M by $\chi_{M^*}(g) = \overline{\chi_M(g)}$. Deduce that $M \cong M^*$ if and only if $\chi_M(g)$ is a real number for every $g \in G$.

    (v) Prove that $g \in G$ is conjugate to g^{-1} if and only if $\chi(g)$ is a real number for every character χ.

19. If $g \in G$ show that $|C_G(g)| = \sum_{i=1}^{r} |\chi_i(g)|^2$. Conclude that the character table gives $|C_G(g)|$ for each $g \in G$.

20. Is the character table

`http://www.chemsoc.org/exemplarchem/entries/2004/ hull_booth/Special_groups/HighSym-Ih.htm`

    (Printout attached) really a character table?

21. Compute the character table of the alternating group A_4.

22. Compute the character table of the dihedral group D_5.

23. Here are some partially completed character tables of some finite groups. I've also listed the orders of some of the conjugacy classes... Use what you know about character tables to fill in the remaining entries.

    (i)

C	C_1	C_2	C_3	C_4	C_5
$\|C\|$	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2					
χ_3	3	-1	0	$\varepsilon^2+\varepsilon^3+1$	$\varepsilon+\varepsilon^4+1$
χ_4		0	1	-1	-1
χ_5				0	

    (Here $\varepsilon = e^{2\pi i/5}$.)

    (ii)

C	C_1	C_2	C_3	C_4	C_5	C_6
$\|C\|$						
χ_1	1	1	1	1	1	
χ_2	1	-1	-1	1	1	
χ_3	1	1	-1	-1	1	
χ_4	1	-1	1	-1	1	
χ_5	2	0	0	1	-1	
χ_6	2	0	0	-1	-1	

(iii)

C	C_1	C_2	C_3	C_4	C_5
$\lvert C\rvert$	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	
χ_4	1	1	-1	-1	
χ_5					

(iv)

C	C_1	C_2	C_3	C_4	C_5
$\lvert C\rvert$	1	6	8	6	3
χ_1	1	1	1	1	1
χ_2		-1		-1	1
χ_3	2			0	
χ_4	3	1			-1
χ_5	3			1	

24. Recall from class how you use the character table of G to find the lattice of normal subgroups of G and their orders. Hence work out which of the groups whose character tables you computed in questions 21–23 are simple groups.

25. (i) Explain how to use the character table of G to find $Z(G)$.

    (ii) Explain how to use the character table of G to determine whether G is nilpotent.

    (iii) Use the character tables to determine which of the groups whose character table you computed in 21–23 are nilpotent groups.

26. Compute the character table of the symmetric group S_6.

    You start frm the permutation characters on singletons, pairs and triplets. It takes a while but you should be able to get

C (123456)	$\lvert C\rvert$	(1)	(12)	(123)	(12)(34)	(1234)	(12)(345)	(12345)	(123)(456)	(12)(3456)	(12)(34)(56)
	1	15	40	45	90	120	144	40	90	15	120
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1	1	1	-1	-1
χ_3	5	3	2	1	1	0	0	-1	-1	-1	-1
χ_4	5	-3	2	1	-1	0	0	-1	-1	1	1
χ_5	9	3	0	1	-1	0	-1	0	1	3	0
χ_6	9	-3	0	1	1	0	-1	0	1	-3	0
χ_7	5	1	-1	1	-1	1	0	2	-1	-3	0
$\chi_8$5	-1	-1	1	1	-1	0	2	-1	3	0	
χ_9	10	-2	1	-2	0	1	0	1	0	2	-1
χ_{10}	10	2	1	-2	0	-1	0	1	0	-2	1
χ_{11}	16	0	-2	0	0	0	1	-2	0	0	0

27. Let G be a finite group, $L_1, \dots, L_r$ be a set of representatives of the isomorphism classes of irreducible $\mathbb{C}G$-modules, $\chi_1, \dots, \chi_r$ be the corresponding irreducible characters.

    (i) Prove that $\dim \mathrm{Hom}_{\mathbb{C}G}(L_i, L_j) = (\chi_i, \chi_j)$.

    (ii) For any finite dimensional $\mathbb{C}G$-modules V and W, prove that $\dim \mathrm{Hom}_{\mathbb{C}G}(V, W) = (\chi_V, \chi_W)$.

28. Let H be a subgroup of a finite group G. Given any $\mathbb{C}H$-module V, let $\mathrm{ind}_H^G V$ denote the $\mathbb{C}G$-module $\mathbb{C}G \otimes_{\mathbb{C}H} V$. Given any $\mathbb{C}G$-module W, let $\mathrm{res}_H^G W$ denote the $\mathbb{C}H$-module obtained from W by restricting the action of G to the subgroup of H.

(i) Prove that

$$\dim \mathrm{Hom}_{\mathbb{C}G}(\mathrm{ind}_H^G V, W) = \dim \mathrm{Hom}_{\mathbb{C}H}(V, \mathrm{res}_H^G W).$$

(ii) Let $t_1, \dots, t_k$ be a set of G/H-coset representatives. Let $v_1, \dots, v_n$ be a basis for V. Prove that $\{t_i \otimes v_j \mid i = 1, \dots, k, j = 1, \dots, n\}$ is a basis for $\mathrm{ind}_H^G V$. How does $g \in G$ act on the basis element $t_i \otimes v_j$?

(iii) Let $\chi \in C(H)$ be the character of the $\mathbb{C}H$-module V. Define $\dot{\chi}(g) = \chi(g)$ if $g \in H$ or 0 if $g \in G - H$. Also let $\mathrm{ind}_H^G \chi \in C(G)$ be the character of the induced module $\mathrm{ind}_H^G V$. Using your answer to (ii), prove that

$$(\mathrm{ind}_H^G \chi)(g) = \sum_{i=1}^{k} \dot{\chi}(t_i^{-1} g t_i).$$

(iv) Let $\psi \in C(G)$ be the character of the $\mathbb{C}G$-module W, and let $\mathrm{res}_H^G \psi \in C(H)$ be its restriction to H. Use the result from problem 27 to prove *Frobenius reciprocity*:

$$(\mathrm{ind}_H^G \chi, \psi)_G = (\chi, \mathrm{res}_H^G \psi)_H$$

where $(.,.)_G$ and $(.,.)_H$ are the usual inner products on $C(G)$ and $C(H)$ respectively.

29. Let H be a subgroup of G. Let χ be the trivial character of H. Let $\mathrm{ind}_H^G \chi$ be the induced character of G like in the previous question. Prove that $\mathrm{ind}_H^G \chi$ is the permutation character arising from the usual action of G by left multiplication on the set G/H of cosets of H in G. Thus induced characters generalize permutation characters...