

# Exercises on chapter 0

1. A partially ordered set (poset) is a set  $X$  together with a relation  $\leq$  such that (a)  $x \leq x$  for all  $x \in X$ ; (b)  $x \leq y$  and  $y \leq x$  implies that  $x = y$  for all  $x, y \in X$ ; (c)  $x \leq y$  and  $y \leq z$  implies that  $x \leq z$  for all  $x, y, z \in X$ .
  - (i) Given a poset  $X$ , define a category  $PO(X)$  as follows: the objects are the elements of the set  $X$ ; for  $x, y \in X$  there is a unique morphism  $x \rightarrow y$  if  $x \leq y$ , otherwise the set of morphisms from  $x$  to  $y$  is the empty set. Work out for yourself how to define the composition of two morphisms (there is only one possible way) and then check that this is indeed a category.
  - (ii) Let  $X$  be the power set of the set  $\{1, 2, 3\}$ , i.e. the set of all subsets of this set, and define a partial order on  $X$  by declaring that  $x \leq y$  if  $x$  is a subset of  $y$ . Make a picture of the category  $PO(X)$  by drawing a vertex for each object and arrow for each morphism. (It is a very simple sort of category because there is at most one arrow between any two objects – usually there will be loads of arrows joining each pair of objects!)
  - (iii) For which posets  $X$  does the category  $PO(X)$  have a zero object?
  - (iv) Prove in the category  $PO(X)$  that  $x \coprod y$  (coproduct) is the least upper bound for  $x$  and  $y$ , and  $x \prod y$  (product) is the greatest lower bound of  $x$  and  $y$ , if they exist.
  - (v) For posets  $X$  and  $Y$ , what does a functor  $F : PO(X) \rightarrow PO(Y)$  mean in the language of posets?
  - (vi) Suppose that  $X$  is a poset and  $\mathbf{ab}$  is the category of abelian groups, and that  $F$  and  $G$  are two functors from  $PO(X)$  to  $\mathbf{ab}$ . What does a natural transformation from  $F$  to  $G$  really mean?
    - (i) To compose  $x \rightarrow y$  and  $y \rightarrow z$  you just joint the arrows to get the unique arrow  $x \rightarrow z$  – which exists by transitivity of the partial order.
    - (ii) You should have 8 vertices, one at the top (the whole set) one at the bottom (the empty set) then singletons and pairs above that.
    - (iii) Only for  $X$  the empty set.
    - (iv) This is the definition. But it takes a little translating – only you can do that!
    - (v) It means an order-preserving map, i.e. a map  $F : X \rightarrow Y$  such that  $x \leq y$  implies that  $F(x) \leq F(y)$ .
    - (vi) Let's work through the definition. For each  $x \in X$ ,  $\eta$  defines a homomorphism of abelian groups  $\eta_x : F(x) \rightarrow G(x)$  such that if  $x \leq y$  then the diagram commutes:

$$\begin{array}{ccc}
 F(x) & \xrightarrow{t} & F(y) \\
 \eta_x \downarrow & & \downarrow \eta_y \\
 F(x) & \xrightarrow{b} & G(y)
 \end{array}$$

where the top and bottom maps  $t$  and  $b$  are the homomorphisms of abelian groups gotten by applying the functors  $F$  and  $G$  to the arrow  $x \rightarrow y$  respectively. The phrase “the diagram commutes” means that the maps  $\eta_y \circ t$  and  $b \circ \eta_x$  are equal in the category of abelian groups!

2. Work in the category of abelian groups. Explain carefully how to make the Cartesian product  $G \times H$  of the underlying sets into the coproduct of  $G$  and  $H$  in the categorical sense. (Usually we write this as  $G \oplus H$  and call it “direct sum” for abelian groups). Give a counterexample to show that if you replace “abelian groups” by “rings” here, the setwise Cartesian product  $R \times S$  of rings  $R$  and  $S$  with the obvious coordinatewise operations cannot in any way be made into the coproduct in the category of rings.

Obviously  $G \times H$  is an abelian group with coordinatewise operation. The maps  $\iota_1 : G \rightarrow G \times H$  and  $\iota_2 : H \rightarrow G \times H$  are the obvious things: in additive notation,  $\iota_1(g) = (g, 0)$  and  $\iota_2(h) = (0, h)$ . Now take another object  $C$  and maps  $\phi_1 : G \rightarrow C$  and  $\phi_2 : H \rightarrow C$ . We need to construct a unique map  $\phi : G \times H \rightarrow C$  such that  $\phi \circ \iota_i = \phi_i$ . But that means for sure that  $\phi(g, 0) = \phi_1(g)$  and  $\phi(0, h) = \phi_2(h)$ . So since we’re in an abelian group we’d better have  $\phi(g, h) = \phi(g, 0) + \phi(0, h) = \phi_1(g) + \phi_2(h)$ . So  $\phi$  is unique – and this map is certainly a morphism of abelian groups which does this job.

If we’re in rings instead, let’s try for an example taking  $R$  and  $S$  both equal to  $\mathbb{Z}$  (the world’s first ring). We want the coproduct to have underlying set  $\mathbb{Z} \times \mathbb{Z}$ , with coordinatewise operations. This is certainly a ring, with identity element  $(1, 1)$ . So our map  $\iota_1 : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$  must send 1 to  $(1, 1)$ , hence  $n$  to  $(n, n)$ . Similarly there is no choice for  $\iota_2$ . Now we’ve got to check if this data satisfies the universal property to be the coproduct. Well, take another object  $D$ , say  $D = \mathbb{Z}$ , and try the identity maps  $\phi_1, \phi_2 : \mathbb{Z} \rightarrow D$ . We need to show that there is a unique map  $\phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $\phi(n, n) = n$  for each integer  $n$ . Well here are two such maps: First map  $(a, b)$  to  $a$ ; Second map  $(a, b)$  to  $b$ . So the uniqueness goes wrong.

3. Consider the category **top** of all topological spaces, i.e. the objects are all topological spaces and the morphisms are all continuous maps between them. What is the product  $X \prod Y$  and the coproduct  $X \coprod Y$  in this category?

The product is the Cartesian product  $X \times Y$  as sets, with open sets being things that can be written as unions of subsets of the form  $U \times V$  for  $U$  open in  $X$  and  $V$  open in  $Y$ . You should remember this as the product topology!

The coproduct is the disjoint union  $X \dot{\cup} Y$  of sets, with open sets being disjoint unions  $U \dot{\cup} V$  for  $U$  open in  $X$  and  $V$  open in  $Y$ .

4. Let  $f : V \rightarrow W$  and  $g : W \rightarrow U$  be linear maps between finite dimensional vector spaces  $W, V$  and  $U$  over some field  $K$ . Let  $f^* : W^* \rightarrow V^*$  and  $g^* : U^* \rightarrow W^*$  be the dual maps. Prove carefully that  $(g \circ f)^* = f^* \circ g^*$ . Hence verify that the duality  $D$  on the category of finite dimensional vector spaces really is a contravariant functor.

Take  $x \in U^*$ , i.e. a linear function on  $U$ . Then,  $(g \circ f)^*(x)$  is an element of  $V^*$ , hence to make sense of it we need to evaluate it on some element  $v \in V$ . We get by the definition of dual map that

$$((g \circ f)^*(x))(v) = x(g(f(v))).$$

Similarly,

$$((f^* \circ g^*)(x))(v) = (f^*(g^*(x)))(v) = (g^*(x))(f(v)) = x(g(f(v))).$$

The two are equal so we’re done.

5. Let  $f : V \rightarrow W$  be a linear map between finite dimensional vector spaces. Fix bases  $v_1, \dots, v_n$  for  $V$  and  $w_1, \dots, w_m$  for  $W$ . Recall the matrix for the linear map  $f$  with respect to these bases is the  $m \times n$  matrix  $A = (a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$  defined from

$$f(v_j) = \sum_{i=1}^m a_{i,j} w_i.$$

- (a) Describe explicitly in terms of the matrix  $A$  how to compute the matrix  $B$  for the linear map  $f^* : W^* \rightarrow V^*$  with respect to the basis  $x_1, \dots, x_m$  for  $W^*$  that is dual to the basis  $w_1, \dots, w_m$  and the basis  $y_1, \dots, y_n$  for  $V^*$  that is dual to the basis  $v_1, \dots, v_n$ .

(b) Prove that  $f$  is injective if and only if  $f^*$  is surjective, and vice versa.

(a) The matrix  $B$  is the transpose of the matrix  $A$ . To check this, we have to show that

$$f^*(x_j) = \sum_{i=1}^n a_{j,i} y_i.$$

Evaluate both sides on  $v_k$ . The left hand side equals  $(f^*(x_j))(v_k) = x_j(f(v_k))$  which is the  $w_j$ -coefficient of  $f(v_k)$  which by the definition of  $A$  is  $a_{j,k}$ . The right hand side equals  $a_{j,k}$  too by definition of dual basis. We're done.

(b) The cheating way is to remember from linear algebra that row rank (number of linearly independent rows) equals column rank (number of linearly independent columns) for a matrix. So by (a) the dimension of the image of  $f$  and of  $f^*$  is equal. Now  $f$  is injective if and only if  $\dim \text{im}(f) = \dim V$ . This is if and only if  $\dim \text{im}(f^*) = \dim V$ , i.e.  $f^*$  is surjective. For the vice versa, apply what we've just proved instead to the map  $f^*$ :  $f^*$  is injective if and only if  $f^{**} = f$  is surjective.

But that was really cheating since we've appealed to a result from linear algebra that we should just prove from scratch. The right statement to prove is that:  $(\text{im } f)^0 = \ker f^*$ . Here for a subspace  $X$  of  $W$ ,  $X^0$  denotes its *annihilator* in  $W^*$ , i.e. the set of all functions in  $W^*$  that are zero on all of  $X$ . Then  $f$  is surjective if and only if  $\text{im } f = W$  if and only if  $(\text{im } f)^0 = W^0 = 0$  if and only if  $\ker f^* = 0$  if and only if  $f^*$  is injective. Similarly (e.g. by taking the same statement with  $f$  replaced by  $f^*$ ) you show that  $(\text{im } f^*)^0 = \ker f$  hence (taking annihilators on both sides)  $\text{im } f^* = (\ker f)^*$ .

To prove this, take  $\alpha \in \ker f^*$ . Then  $\alpha(f(v)) = (f^*(\alpha))(v) = 0$  for all  $v \in V$ . Hence,  $\alpha$  annihilates  $\text{im } f$ . We've shown  $\ker f^* \subseteq (\text{im } f)^0$ .

Conversely, take  $\alpha \in (\text{im } f)^0$ . Then  $(f^*(\alpha))(v) = \alpha(f(v)) = 0$  for all  $v \in V$ . Hence  $f^*(\alpha) = 0$ . Hence  $\alpha \in \ker f^*$ . We've shown  $(\text{im } f)^0 \subseteq \ker f^*$ .

This question is well worth spending extra time to master. Things involving dual maps are easy to get confused about – you need to practice enough not to get muddled up with which space you are working in!

6. Remember the category  $PO(X)$  you sketched in 1(ii). Consider instead the category  $\mathcal{C}$  with just 4 objects,  $a, b, c, d$  and a unique morphism from  $a$  to  $a, b, c$  and  $d$ , a unique morphism from  $b$  to  $b, c$  and  $d$ , a unique morphism from  $c$  to  $c$  and  $d$ , and a unique morphism from  $d$  to  $d$ . Draw the picture!

(a) Check that there is a functor  $F$  from  $\mathcal{C}$  to  $PO(X)$  such that  $F(a) = \emptyset, F(b) = \{1\}, F(c) = \{1, 2\}$  and  $F(d) = \{1, 2, 3\}$ .

(b) Check that there is a functor  $G$  from  $PO(X)$  to  $\mathcal{C}$  such that

$$\begin{aligned} F(\emptyset) &= a, \\ F(\text{any one element subset}) &= b, \\ F(\text{any two element subset}) &= c \\ F(\{1, 2, 3\}) &= d. \end{aligned}$$

(c) Prove that the functor  $G \circ F$  is naturally isomorphic to the identity functor on the category  $\mathcal{C}$ .

(d) Is the functor  $F \circ G$  naturally isomorphic to the identity functor on the category  $PO(X)$ ?

(a),(b) Obvious. Make sure you actually bothered to define  $F$  and  $G$  on morphisms though: it is not enough to define a functor just to say what it does on objects!!! But in this case there really is no choice since there is at most one arrow between each pair of objects...

(c)  $G \circ F$  is just the functor mapping  $a$  to  $a$ ,  $b$  to  $b$ ,  $c$  to  $c$  and  $d$  to  $d$ . So it is EQUAL to the identity functor. That's even better than being naturally isomorphic to it.

(d) Suppose there is a natural isomorphism  $\eta : F \circ G \rightarrow \text{id}$ . So for each object  $x$  we need a morphism from  $F \circ G(x)$  to  $x$ . For example if  $x = \{2, 3\}$  we need a morphism from  $F \circ G(x) = \{1, 2\}$  to  $\{2, 3\}$ . But there is no such arrow! So NO.

7. A *group scheme* is a functor from the category of commutative rings to the category of groups.

(i) Here are three group schemes:  $GL_n$ ,  $SL_n$  and  $\mu$ . I'll define them for you on objects: for a commutative ring  $R$ ,  $GL_n(R)$  is the group of all invertible  $n \times n$  matrices with entries in the ring  $R$  under matrix multiplication,  $SL_n(R)$  is the subgroup of  $GL_n(R)$  consisting of all matrices of determinant 1, and  $\mu(R)$  is the group  $R^\times$  of all units in  $R$ . Now work out how to define them on a ring homomorphism  $f : R \rightarrow S$  to make these into group schemes for yourself.

(ii) A morphism of group schemes means the same thing as a natural transformation of functors. For group schemes  $G$  and  $H$ , write out in words exactly what a morphism  $\eta : G \rightarrow H$  of group schemes is really saying.

(iii) Check that the map  $\eta : SL_n \rightarrow GL_n$  defined by letting  $\eta_R : SL_n(R) \rightarrow GL_n(R)$  be the natural inclusion for each commutative ring  $R$  is a morphism of group schemes. (In this case  $SL_n$  is a *subgroup* of  $GL_n$ .)

(iv) Check that the map  $\eta : GL_n \rightarrow \mu$  defined by letting  $\eta_R : GL_n(R) \rightarrow R^\times$  be the map sending a matrix  $g \in GL_n(R)$  to its determinant is a morphism of group schemes. (In this case it makes sense to write  $\ker \eta = SL_n$ .)

(i) Define  $GL_n(f) : GL_n(R) \rightarrow GL_n(S)$  to be the map obtained by applying  $f$  to each of the entries of a matrix  $g \in GL_n(R)$ . Similarly for  $SL_n(f)$ . Finally,  $\mu(f) : R^\times \rightarrow S^\times$  is just the restriction of the function  $f$  to the units in  $R$ .

(ii) It means for each commutative ring  $R$  a group homomorphism  $\eta_R : G(R) \rightarrow H(R)$  such that for every arrow  $f : R \rightarrow S$  we have that  $\eta_S \circ G(f) = H(f) \circ \eta_R$ .

(iii) You just have to observe that a ring homomorphism  $f$  applied to the entries of a matrix of determinant 1 still gives a matrix of determinant 1.

(iv) First note that determinant is a group homomorphism  $GL_n(R) \rightarrow R^\times$ , i.e. it is multiplicative. Now if we have  $g \in GL_n(R)$  and a ring homomorphism  $f : R \rightarrow S$  we need to see that  $\det(f(g)) = f(\det(g))$ . This follows by the definition of ring homomorphism.

8. This exercise is more about dual spaces. Let  $K$  be a field, and recall that for every finite dimensional vector space  $V$  there is an isomorphism  $i_V : V \rightarrow V^{**}$  defined in class. That is,  $i : \text{Id}_V \rightarrow D \circ D$  is a natural isomorphism between the identity functor and the double dual.

(i) For a subspace  $U$  of  $V$ , the *annihilator*  $U^0$  is defined to be  $\{f \in V^* \mid f(u) = 0 \text{ for all } u \in U\}$ . Show that restriction of functions induces an isomorphism of vector spaces  $V^*/U^0 \rightarrow U^*$ . Deduce that  $\dim U^0 = \dim V - \dim U$ .

(ii) Prove that the map  $i_V : V \rightarrow V^{**}$  maps a subspace  $U$  of  $V$  isomorphically onto the subspace  $U^{00}$  of  $V^{**}$ .

(i) Define a linear map  $\theta : V^* \rightarrow U^*$  mapping a function  $f : V \rightarrow K$  to its restriction  $f|_U : U \rightarrow K$ . So  $\theta$  is just "restriction of functions". Let's compute its kernel. We have that  $f \in \ker \theta$  if and only if  $f|_U = 0$ , i.e.  $f \in U^0$ . Hence,  $\ker \theta = U^0$ .

So by the universal property of quotients,  $\theta$  induces an injective map  $\bar{\theta} : V^*/U^0 \rightarrow U^*$ . Finally we prove that this is onto. Take any element  $f \in U^*$ . Pick a basis  $u_1, \dots, u_m$  for  $U$  and extend to a basis  $u_1, \dots, u_m, u_{m+1}, \dots, u_n$  for  $V$ . Now define  $\hat{f} \in V^*$  by setting  $\hat{f}(u_i) = f(u_i)$  for  $i = 1, \dots, m$  and  $\hat{f}(u_i) = 0$  for  $i = m+1, \dots, n$ . Then  $\hat{f} \in V^*$  is a function which restricts to  $f \in U^*$ . Hence  $\theta$  is onto.

So in particular  $\dim V^*/\dim U^0 = \dim V - \dim U^0 = \dim U$ .

(ii) Here  $U^{00}$  is the subspace of  $V^{**}$  consisting of all functions on  $V^*$  which annihilate  $U^0$ . Let's show that  $i_V(U) \subseteq U^{00}$ . Take  $u \in U$  and  $f \in U^0$ . We need to show that  $(i_V(u))(f) = 0$ . But by definition of  $i_V$  that's  $\hat{u}(f) = f(u)$  which is zero as  $f \in U^0$ .

Since we already know  $i_V$  is injective on all of  $V$  it just remains to check that  $U$  and  $U^{00}$  have the same dimension. For that use (i).

9. This exercise is more linear algebra, about bilinear forms. For a finite dimensional vector space  $V$ , a *bilinear form* on  $V$  is a  $K$ -bilinear map  $(\cdot, \cdot) : V \times V \rightarrow K$ . Given a basis  $v_1, \dots, v_n$  for  $V$ , the *matrix* of the bilinear form  $(\cdot, \cdot)$  with respect to this basis is the matrix  $(a_{i,j})_{1 \leq i,j \leq n}$  defined from  $a_{i,j} = (v_i, v_j)$ . For example, for the space  $\mathbb{R}^n$ , the matrix of the usual dot product in the standard basis is just the identity matrix.

(i) Suppose that  $v_1, \dots, v_n$  and  $w_1, \dots, w_n$  are two different basis. Let  $w_j = \sum_{i=1}^n p_{i,j} v_i$ , so  $P = (p_{i,j})_{1 \leq i,j \leq n}$  is the transition matrix from the  $v$ -basis to the  $w$ -basis. Let  $A$  be the matrix of the given bilinear form  $(\cdot, \cdot)$  in the  $v$ -basis and  $B$  be the matrix of  $(\cdot, \cdot)$  in the  $w$ -basis. Express  $B$  in terms of  $A$  and  $P$ .

(ii) We will from now on only ever talk about *symmetric* bilinear forms or *skew-symmetric* bilinear forms, i.e. ones with  $(v, w) = (w, v)$  or  $(v, w) = -(w, v)$  for all  $v, w \in V$ , respectively. What does the matrix of a symmetric bilinear form look like? A skew-symmetric one?

(iii) Let  $v_1, \dots, v_n$  be a basis for  $V$  and let  $A$  be the matrix of the given bilinear form  $(\cdot, \cdot)$  in this basis. Let  $v$  and  $w$  be two vectors in  $V$ , written as column vectors  $[v]$  and  $[w]$  in terms of the given basis. Prove that  $(v, w) = [v]^T A [w]$ .

(i) We have that  $b_{i,j} = (w_i, w_j) = \sum_{k,l} p_{k,i} p_{l,j} (v_k, v_l)$ . This says that  $b_{i,j} = \sum_{k,l} p_{k,i} a_{k,l} p_{l,j}$ , hence  $B = P^T A P$ . This is the change of basis formula for bilinear forms.

(ii) The matrix for a symmetric bilinear form satisfies  $A^T = A$ ; for a skew-symmetric form  $A^T = -A$ .

(iii) If  $v = \sum a_i v_i$  and  $w = \sum b_j v_j$ , then  $[v]$  is the column vector with entries  $a_1, \dots, a_n$  from top to bottom and  $[w]$  is the column vector with entries  $b_1, \dots, b_n$ . By definition of bilinear form,  $(v, w) = \sum_{i,j} a_i b_j (v_i, v_j) = \sum_{i,j} a_i a_{i,j} b_j$ . This is the definition of the matrix product  $[v]^T A [w]$ .

10. Let  $(\cdot, \cdot)$  be either a symmetric or a skew-symmetric bilinear form on a finite dimensional vector space  $V$ . Given a subspace  $U$  of  $V$ , let  $U^\perp$  denote the subspace  $\{v \in V \mid (u, v) = 0 \text{ for all } u \in U\}$ . (For not necessarily symmetric/skew-symmetric forms, this could be called the “right perp” then there would be an analogous notion of “left perp” with  $(u, v)$  replaced by  $(v, u)$  – but the good thing about symmetric/skew symmetric forms is the left and right perps mean the same thing!)

(i) Define a map  $\theta : V \rightarrow V^*$  by mapping  $v \in V$  to the unique function  $f_v \in V^*$  with  $f_v(w) = (v, w)$  for all  $w \in V$ . Prove that this map is an isomorphism if and only if the form is non-degenerate, meaning that its *radical*  $V^\perp = \{v \in V \mid (u, v) = 0 \text{ for all } u \in V\}$  is zero.

(ii) Prove that  $(\cdot, \cdot)$  is non-degenerate if and only if its matrix with respect to some basis is invertible, i.e. has non-zero determinant.

(iii) Suppose that the form is non-degenerate and that  $U$  is a subspace of  $V$ . Prove that the map  $\theta$  from (i) maps  $U^\perp$  isomorphically onto  $U^0$ . Deduce that  $\dim V = \dim U + \dim U^\perp$ . Hence if  $U$  is a *non-degenerate subspace* of  $V$ , meaning that the bilinear form  $U \times U \rightarrow K$  obtained by restricting the form on  $V$  to the subspace  $U$  is non-degenerate, show that  $V = U \oplus U^\perp$ . In this case,  $U^\perp$  is called the *orthogonal complement* of the non-degenerate subspace  $U$ .

(iv) Show that  $U \subseteq (U^\perp)^\perp$  with equality if the form is non-degenerate.

(v) Finally let’s improve on (iii) in the case the form is degenerate. Explain how the form on  $V$  induces a non-degenerate form on the quotient vector space  $V/V^\perp$ . Using this trick, prove in general that  $\dim V = \dim U + \dim U^\perp - \dim(U \cap V^\perp)$ .

(i) By dimension, we just need to check that  $\ker \theta = 0$  if and only if  $V^\perp = 0$ . In fact we’ll show that  $\ker \theta = V^\perp$ :  $v \in \ker \theta$  if and only if  $f_v = 0$  if and only if  $f_v(u) = 0$  for all  $u \in V$  if and only if  $(v, u) = 0$  for all  $u \in V$  if and only if  $v \in V^\perp$ .

(ii) Suppose that  $v \in V^\perp$ . Then,  $(u, v) = 0$  for all  $u \in V$ . Pick a basis for  $V$  and let  $A$  be the resulting matrix of the bilinear form,  $[u]$  be the column vector representing  $u$  and  $[v]$  representing  $v$ . Then,  $[u]^T A [v] = 0$  for all column vectors  $u$ . Hence  $A [v] = 0$ .

It follows that  $V^\perp = 0$  if and only if the equation  $A[v] = 0$  has a unique solution, i.e. the matrix  $A$  is invertible.

(iii) Take  $v \in V$ . We have that  $f_v(u) = 0$  for all  $u \in U$  if and only if  $(u, v) = 0$  for all  $u \in U$ . This shows that  $f_v \in U^0$  if and only if  $v \in U^\perp$ . Hence  $\theta$  (which is already an isomorphism by (i) as the form is non-degenerate) maps  $U^\perp$  isomorphically onto  $U^0$ . So since by question 7 we know that  $\dim U^0 = \dim V - \dim U$  we get that  $\dim V = \dim U + \dim U^\perp$  too.

(iv) Let's show that  $U \subseteq (U^\perp)^\perp$ . Then you get equality in the non-degenerate case by dimension. We need to see that  $(u, v) = 0$  for all  $u \in U$  and  $v \in U^\perp$ . But that's obvious.

(v) Remember the quotient  $V/V^\perp$  consists of cosets  $v+V^\perp$ . Define a form on it by  $(u+V^\perp, v+V^\perp) := (u, v)$ . Is this well-defined? Yes, because of the definition of  $V^\perp$ . Is it non-degenerate? Say  $(u+V^\perp, v+V^\perp) = 0$  for all  $v \in V$ . Then,  $(u, v) = 0$  for all  $v \in V$ . Hence  $u \in V^\perp$ . Hence  $u+V^\perp = 0$ .

Now we apply (iii) to the subspace  $(U+V^\perp)/V^\perp$  of  $V/V^\perp$ . Its perp is  $(U^\perp+V^\perp)/V^\perp$  which is simply  $U^\perp/V^\perp$ . So we get that

$$\dim(V/V^\perp) = \dim(U+V^\perp)/V^\perp + \dim(U^\perp/V^\perp).$$

The left hand side is  $\dim V - \dim V^\perp$ . And since  $(U+V^\perp)/V^\perp \cong U/(U \cap V^\perp)$  by the second isomorphism theorem, the right hand side is  $\dim U - \dim(U \cap V^\perp) + \dim U^\perp - \dim V^\perp$ . We're done.

11. This exercise is about skew-symmetric bilinear forms. Let  $(\cdot, \cdot)$  be a skew-symmetric bilinear form on a finite dimensional vector space  $V$  over a field  $K$  of characteristic different from 2. The goal is to prove that  $V$  possesses a basis  $v_1, \dots, v_{2m}, v_{2m+1}, \dots, v_n$  with respect to which the matrix of the form looks like  $\text{diag}(J, \dots, J, 0, \dots, 0)$  ( $m$   $J$ 's and  $(n-2m)$  0's) where  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . This is the *canonical form* for a skew-symmetric form.

(i) If the form is zero on all of  $V$  we are done already.

(ii) So assume there are vectors  $v_1, v_2$  with  $(v_1, v_2) \neq 0$ . Explain why  $U = \text{span}\{v_1, v_2\}$  is 2-dimensional and that you can rescale  $v_2$  so that the matrix of the restriction of the form to  $U$  in the basis  $v_1, v_2$  is  $J$ .

(iii) Why is  $V = U \oplus U^\perp$ ?

(iv) Now use induction to pick a basis for  $U^\perp$  and complete the construction.

(v) An isotropic subspace of  $V$  means a subspace such that the restriction of the form to this subspace is zero. What is the dimension of a maximal totally isotropic subspace of  $V$  in terms of  $m$  and  $n$ ?

(vi) Prove that if the form  $(\cdot, \cdot)$  is non-degenerate then  $\dim V$  is even. In that case  $V$  is called a symplectic vector space and the form is called a symplectic form...

(i) Sure. Any basis will do.

(ii) Since  $(v_1, v_1) = -(v_1, v_1)$  and the field is not of characteristic 2, we have that  $(v_1, v_1) = 0$ . So  $v_2$  cannot be a scalar multiple of  $v_1$  since  $(v_1, v_2) \neq 0$ . Hence  $U$  is 2-dimensional. Clearly we can replace  $v_2$  by  $v_2/(v_1, v_2)$  if necessary to reduce to the situation that  $(v_1, v_2) = 1$ . Then the matrix is  $J$ .

(iii) Note that  $U \cap U^\perp = 0$  since the restriction of the form to  $U$  has matrix  $J$  which is non-degenerate. Hence  $U \cap V^\perp = 0$  too. So by 9(v) we get that  $\dim V = \dim U + \dim U^\perp$ . Now  $U \cap U^\perp = 0$  and  $\dim U + \dim U^\perp = \dim V$  together imply that  $V = U \oplus U^\perp$ .

(iv) Yes that does the trick!

(v)  $n - m$ : the subspace spanned by  $v_1, v_3, \dots, v_{2m-1}, v_{2m+1}, v_{2m+2}, \dots, v_n$  is a maximal isotropic subspace.

(vi) The rank of the matrix of the form is  $2m$ , so its non-degenerate if and only if  $\dim V = 2m$ .