Exercises on chapter 0

1. A partially ordered set (poset) is a set X together with a relation $\leq$ such that (a) $x \leq x$ for all $x \in X$; (b) $x \leq y$ and $y \leq x$ implies that $x = y$ for all $x, y \in X$; (c) $x \leq y$ and $y \leq z$ implies that $x \leq z$ for all $x, y, z \in X$.

(i) Given a poset X, define a category $PO(X)$ as follows: the objects are the elements of the set X; for $x, y \in X$ there is a unique morphism $x \to y$ if $x \leq y$, otherwise the set of morphisms from x to y is the empty set. Work out for yourself how to define the composition of two morphisms (there is only one possible way) and then check that this is indeed a category.

(ii) Let X be the power set of the set $\{1, 2, 3\}$, i.e. the set of all subsets of this set, and define a partial order on X by declaring that $x \leq y$ if x is a subset of y. Make a picture of the category $PO(X)$ by drawing a vertex for each object and arrow for each morphism. (It is a very simple sort of category because there is at most one arrow between any two objects – usually there will be loads of arrows joining each pair of objects!)

(iii) For which posets X does the category $PO(X)$ have a zero object?

(iv) Prove in the category $PO(X)$ that $x \coprod y$ (coproduct) is the least upper bound for x and y, and $x \prod y$ (product) is the greatest lower bound of x and y, if they exist.

(v) For posets X and Y, what does a functor $F : PO(X) \to PO(Y)$ mean in the language of posets?

(vi) Suppose that X is a poset and $\mathfrak{ab}$ is the category of abelian groups, and that F and G are two functors from $PO(X)$ to $\mathfrak{ab}$. What does a natural transformation from F to G really mean?

2. Work in the category of abelian groups. Explain carefully how to make the Cartesian product $G \times H$ of the underlying sets into the coproduct of G and H in the categorical sense. (Usually we write this as $G \oplus H$ and call it "direct sum" for abelian groups). Give a counterexample to show that if you replace "abelian groups" by "rings" here, the setwise Cartesian product $R \times S$ of rings R and S with the obvious coordinatewise operations cannot in any way be made into the coproduct in the category of rings.

3. Consider the category $\mathfrak{top}$ of all topological spaces, i.e. the objects are all topological spaces and the morphisms are all continuous maps between them. What is the product $X \prod Y$ and the coproduct $X \coprod Y$ in this category?

4. Let $f : V \to W$ and $g : W \to U$ be linear maps between finite dimensional vector spaces W, V and U over some field K. Let $f^* : W^* \to V^*$ and $g^* : U^* \to W^*$ be the dual maps. Prove carefully that $(g \circ f)^* = f^* \circ g^*$. Hence verify that the duality D on the category of finite dimensional vector spaces really is a contravariant functor.

5. Let $f : V \to W$ be a linear map between finite dimensional vector spaces. Fix bases $v_1, \ldots, v_n$ for V and $w_1, \ldots, w_m$ for W. Recall the matrix for the linear map f with respect to these bases is the $m \times n$ matrix $A = (a_{i,j})_{1\le i\le m, 1\le j\le n}$ defined from

$$f(v_j) = \sum_{i=1}^{m} a_{i,j} w_i.$$

(a) Describe explicitly in terms of the matrix A how to compute the matrix B for the linear map $f^* : W^* \to V^*$ with respect to the basis $x_1, \ldots, x_m$ for W^* that is dual to the basis $w_1, \ldots, w_m$ and the basis $y_1, \ldots, y_n$ for V^* that is dual to the basis $v_1, \ldots, v_n$.

(b) Prove that f is injective if and only if f^* is surjective, and vice versa.

6. Remember the category $PO(X)$ you sketched in 1(ii). Consider instead the category $\mathscr{C}$ with just 4 objects, a, b, c, d and a unique morphism from a to a, b, c and d, a unique morphism from b to b, c and d, a unique morphism from c to c and d, and a unique morphism from d to d. Draw the picture!

   (a) Check that there is a functor F from $\mathscr{C}$ to $PO(X)$ such that $F(a) = \varnothing, F(b) = \{1\}$, $F(c) = \{1, 2\}$ and $F(d) = \{1, 2, 3\}$.

   (b) Check that there is a functor G from $PO(X)$ to $\mathscr{C}$ such that

$$\begin{aligned} F(\varnothing) &= a, \\ F(\text{any one element subset}) &= b, \\ F(\text{any two element subset}) &= c \\ F(\{1, 2, 3\}) &= d. \end{aligned}$$

   (c) Prove that the functor $G \circ F$ is naturally isomorphic to the identity functor on the category $\mathscr{C}$.

   (d) Is the functor $F \circ G$ naturally isomorphic to the identity functor on the category $PO(X)$?

7. A *group scheme* is a functor from the category of commutative rings to the category of groups.

   (i) Here are three group schemes: GL_n, SL_n and μ. I'll define them for you on objects: for a commutative ring R, $GL_n(R)$ is the group of all invertible $n \times n$ matrices with entries in the ring R under matrix multiplication, $SL_n(R)$ is the subgroup of $GL_n(R)$ consisting of all matrices of determinant 1, and $\mu(R)$ is the group $R^\times$ of all units in R. Now work out how to define them on a ring homomorphism $f : R \to S$ to make these into group schemes for yourself.

   (ii) A morphism of group schemes means the same thing as a natural transformation of functors. For group schemes G and H, write out in words exactly what a morphism $\eta : G \to H$ of group schemes is really saying.

   (iii) Check that the map $\eta : SL_n \to GL_n$ defined by letting $\eta_R : SL_n(R) \to GL_n(R)$ be the natural inclusion for each commutative ring R is a morphism of group schemes. (In this case SL_n is a *subgroup* of GL_n.)

   (iv) Check that the map $\eta : GL_n \to \mu$ defined by letting $\eta_R : GL_n(R) \to R^\times$ be the map sending a matrix $g \in GL_n(R)$ to its determinant is a morphism of group schemes. (In this case it makes sense to write $\ker \eta = SL_n$).

8. This exercise is more about dual spaces. Let K be a field, and recall that for every finite dimensional vector space V there is an isomorphism $i_V : V \to V^{**}$ defined in class. That is, $i : \mathrm{Id}_V \to D \circ D$ is a natural isomorphism between the identity functor and the double dual.

(i) For a subspace U of V, the *annihilator* U^0 is defined to be $\{f \in V^* | f(u) = 0 \text{ for all } u \in U\}$. Show that restriction of functions induces an isomorphism of vector spaces $V^*/U^0 \to U^*$. Deduce that $\dim U^0 = \dim V - \dim U$.

(ii) Prove that the map $i_V : V \to V^{**}$ maps a subspace U of V isomorphically onto the subspace U^{00} of V^{**}.

9. This exercise is more linear algebra, about bilinear forms. For a finite dimensional vector space V, a *bilinear form* on V is a K-bilinear map $(.,.) : V \times V \to K$. Given a basis $v_1, \dots, v_n$ for V, the *matrix* of the bilinear form $(.,.)$ with respect to this basis is the matrix $(a_{i,j})_{1 \le i,j \le n}$ defined from $a_{i,j} = (v_i, v_j)$. For example, for the space $\mathbb{R}^n$, the matrix of the usual dot product in the standard basis is just the identity matrix.

   (i) Suppose that $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are two different basis. Let $w_j = \sum_{j=1}^n p_{i,j} v_i$, so $P = (p_{i,j})_{1 \le i,j \le n}$ is the transition matrix from the v-basis to the w-basis. Let A be the matrix of the given bilinear form $(.,.)$ in the v-basis and B be the matrix of $(.,.)$ in the w-basis. Express B in terms of A and P.

   (ii) We will from now on only ever talk about *symmetric* bilinear forms or *skew-symmetric* bilinear forms, i.e. ones with $(v, w) = (w, v)$ or $(v, w) = -(w, v)$ for all $v, w \in V$, respectively. What does the matrix of a symmetric bilinear form look like? A skew-symmetric one?

   (iii) Let $v_1, \dots, v_n$ be a basis for V and let A be the matrix of the given bilinear form $(.,.)$ in this basis. Let v and w be two vectors in V, written as column vectors $[v]$ and $[w]$ in terms of the given basis. Prove that $(v, w) = [v]^T A[w]$.

10. Let $(.,.)$ be either a symmetric or a skew-symmetric bilinear form on a finite dimensional vector space V. Given a subspace U of V, let $U^\perp$ denote the subspace $\{v \in V \mid (u, v) = 0 \text{ for all } u \in U\}$. (For not necessarily symmetric/skew-symmetric forms, this could be called the "right perp" then there would be an analogous notion of "left perp" with (u, v) replaced by (v, u) – but the good thing about symmetric/skew symmetric forms is the left and right perps mean the same thing!)

    (i) Define a map $\theta : V \to V^*$ by mapping $v \in V$ to the unique function $f_v \in V^*$ with $f_v(w) = (v, w)$ for all $w \in V$. Prove that this map is an isomorphism if and only if the form is non-degenerate, meaning that its *radical* $V^\perp = \{v \in V \mid (u, v) = 0 \text{ for all } u \in V\}$ is zero.

    (ii) Prove that $(.,.)$ is non-degenerate if and only if its matrix with respect to some basis is invertible, i.e. has non-zero determinant.

    (iii) Suppose that the form is non-degenerate and that U is a subspace of V. Prove that the map θ from (i) maps $U^\perp$ isomorphically onto U^0. Deduce that $\dim V = \dim U + \dim U^\perp$. Hence if U is a *non-degenerate subspace* of V, meaning that the bilinear form $U \times U \to K$ obtained by restricting the form on V to the subspace U is non-degenerate, show that $V = U \oplus U^\perp$. In this case, $U^\perp$ is called the *orthogonal complement* of the non-degenerate subspace U.

    (iv) Show that $U \subseteq (U^\perp)^\perp$ with equality if the form is non-degenerate.

    (v) Finally let's improve on (iii) in the case the form is degenerate. Explain how the form on V induces a non-degenerate form on the quotient vector space $V/V^\perp$. Using this trick, prove in general that $\dim V = \dim U + \dim U^\perp - \dim(U \cap V^\perp)$.

11. This exercise is about skew-symmetric bilinear forms. Let $(.,.)$ be a skew-symmetric bilinear form on a finite dimensional vector space V over a field K of characteristic different from 2. The goal is to prove that V possesses a basis $v_1, \dots, v_{2m}, v_{2m+1}, \dots, v_n$ with respect to which the matrix of the form looks like $\operatorname{diag}(J, \dots, J, 0, \dots, 0)$ (m J's and $(n-2m)$ 0's) where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This is the *canonical form* for a skew-symmetric form.

    (i) If the form is zero on all of V we are done already.

(ii) So assume there are vectors v_1, v_2 with $(v_1, v_2) \neq 0$. Explain why $U = span\{v_1, v_2\}$ is 2-dimensional and that you can rescale v_2 so that the matrix of the restriction of the form to U in the basis v_1, v_2 is J.

(iii) Why is $V = U \oplus U^{\perp}$?

(iv) Now use induction to pick a basis for $U^{\perp}$ and complete the construction.

(v) An isotropic subspace of V means a subspace such that the restriction of the form to this subspace is zero. What is the dimension of a maximal totally isotropic subspace of V in terms of m and n?

(vi) Prove that if the form $(.,.)$ is non-degenerate then $\dim V$ is even. In that case V is called a symplectic vector space and the form is called a symplectic form...