Chapter 0

Categories

Throughout the course, I will try to include a little category theory and a little linear algebra as we need it. I think I'll start off in this preliminary chapter with the basic language of category theory as it is a good conceptual way of thinking of abstract algebra: the study of the objects in the basic algebraic categories.

0.1 Categories

There are some logical problems with the foundations of category theory. So we're going to need some basic notions of set theory, about which I want to be as vague as possible. In set theory, there are two basic notions: the notion of a class and the notion of 'is an element of', $\in$. It's best not to try to define these too carefully or you start running into paradoxes!

A *set* is then a "small" class: formally, a set is defined to be a class that is an element of another class. Then there are some axioms giving you existence of various sets, such as the empty set, the power set of a set, unions, intersections, and so on. These axioms guarantee that there are at least enough sets to do anything reasonable with. Most of these axioms do not apply to classes though, preventing paradoxes like the 'set of all sets' (its a class!).

Now, a *category* $\mathscr{C}$ consists of a *class* $\mathrm{ob}(\mathscr{C})$ of *objects* and for each pair A, B of objects, a *set* $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of *morphisms* from A to B. (In case the class $\mathrm{ob}(\mathscr{C})$ is actually itself a set, the category is called *small*.) We write simply $f : A \to B$ to indicate that f is a morphism from A to B. So you should think of a morphism as simply an 'arrow' from A to B – indeed I'll often call a morphism an arrow to emphasize that part of the data of a morphism is the name of its source object and its destination object... Moreover, there is a given map

$$\mathrm{Hom}_{\mathscr{C}}(B, C) \times \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$$

for each triple of objects A, B, C. We write the image of a pair (f, g) of morphisms under this map by $f \circ g$ and call it the *composition* of f and g. (Note wherever possible I write maps on the left.) In addition, the following axioms are imposed:

(C1) for every object A, there exists a distinguished arrow $\mathrm{id}_A : A \to A$ such that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$ for all other arrows $f : B \to A$ and $g : A \to B$.

(C2) given arrows $f : A \to B, g : B \to C, h : C \to D$, we have that $h \circ (g \circ f) = (h \circ g) \circ f$.

For obvious reasons, (C1) is called the 'identity axiom' and (C2) is called the 'associative axiom'. Here's an immediate consequence of the axioms: if $i : A \to A$ is an arrow such that $i \circ f = f$ and $g \circ i = g$ for every $f : B \to A$ and $g : A \to B$, then $i = \mathrm{id}_A$. In other words, the identity morphism id_A is *unique*. Proof: $i = \mathrm{id}_A \circ i = \mathrm{id}_A$.

In algebra, we will study various sorts of algebraic structure. The notion of category is the umbrella to keep the various structures under! A few more definitions.

We say $\mathscr{B}$ is a *subcategory* of $\mathscr{C}$ if $\mathrm{ob}(\mathscr{B}) \subseteq \mathrm{ob}(\mathscr{C})$ and for every pair of objects A, B in $\mathscr{B}$, $\mathrm{Hom}_{\mathscr{B}}(A, B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A, B)$. A subcategory $\mathscr{B}$ of $\mathscr{C}$ is called a *full subcategory* if in fact $\mathrm{Hom}_{\mathscr{B}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$ for every pair of objects A, B in $\mathscr{B}$. Note to specify a full subcategory of $\mathscr{C}$ it is sufficient just to specify a subclass of the objects of $\mathscr{C}$: then one has no choice but to take all morphisms that make sense to obtain the morphisms in the subcategory.

A morphism $f : A \to B$ in a category $\mathscr{C}$ is called an *isomorphism* if there exists another morphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

0.2 Examples of categories

So I now want to define the various categories that we're going to be working with.

0.2.1. The category $\mathfrak{sets}$ of sets. The objects are all sets. The morphisms are just the functions between the sets. Recall a function $f : A \to B$ between two sets means a subset f of $A \times B = \{(a, b) \mid a \in A, b \in B\}$ such that for *every* $a \in A$ there exists a *unique* $b \in B$ with $(a, b) \in f$ (of course we always write $f(a) = b$ instead of $(a, b) \in f$!) For example, the empty set $\varnothing$ is a set, hence an object in the category of sets. Given any other set A, $\mathrm{Hom}_{\mathfrak{sets}}(\varnothing, A)$ consists of exactly one morphism – namely the function which in the set notation for functions is just the empty set!

0.2.2. The category $\mathfrak{mon}$ of monoids. A *monoid* is a set S with an associative operation $S \times S \to S$ (i.e. writing the operation just as a product, $a(bc) = (ab)c$ for all $a, b, c \in S$) such that there exists a (necessarily unique) element $1 \in S$ satisfying $1a = a1 = a$ for all $a \in S$. This element 1 is called the *identity element*. Note the associativity means that we can unambiguously write $a_1 a_2 \dots a_n$ for the product of n elements of S: any way of interpreting this by putting brackets in gives the same outcome by associativity (*Warning:* this is harder to prove formally than you might think!).

Given a monoid S, I can associate to S a category consisting of just one object $*$ and with the set of morphisms $* \to *$ being precisely the set S underlying the monoid. The rule for composition of morphisms is just the multiplication in the monoid S. So: a monoid is exactly the same thing as a *category with one element*. Turning this around, you can think of a category as a generalization of a monoid! Note the argument given in the previous section to prove that the identity morphism id_* is unique proves at the same time that the identity element 1 of a monoid S is uniquely determined.

I haven't yet defined the category of monoids: I haven't told you the set of morphisms $\mathrm{Hom}_{\mathfrak{mon}}(S, T)$ between two monoids S and T. By definition, a morphism $f : S \to T$ is a function such that $f(ab) = f(a)f(b)$ for all $a, b \in S$ and such that $f(1_S) = 1_T$. Note the second axiom of a morphism here is not a consequence of the first!! Now you can check that with this definition of morphism, we do indeed obtain a category...

0.2.3. The category $\mathfrak{groups}$ of groups. A group is a monoid G such that in addition, for every element $g \in G$ there is a *two-sided inverse* $g^{-1} \in G$, i.e. an element such that $g^{-1}g = 1 = gg^{-1}$. Note the inverse g^{-1} of g is uniquely determined: given another element g' with the same property $g'g = 1 = gg'$, we have that

$$g^{-1} = 1g^{-1} = (g'g)g^{-1} = g'(gg^{-1}) = g'1 = g'.$$

A morphism of groups is the same as a morphism of monoids, and this gives us the category of groups. In fact, the category of groups is a full subcategory of the category of monoids (because morphisms mean the same thing and groups are just special monoids).

You should notice that to test if a function $f : G \to H$ is a morphism of groups, henceforth called a *homomorphism*, it suffices to check the multiplicative property that $f(g_1 g_2) = f(g_1)f(g_2)$. The other condition that $f(1_G) = 1_H$ is automatic. To see this, first observe that if $gh = h$ for *any* $g, h \in G$ then $g = 1$. Indeed, $gh = h$ implies $g = ghh^{-1} = hh^{-1} = 1$. Hence, since $f(1_G)f(1_G) = f(1_G 1_G) = f(1_G)$, we deduce that $f(1_G) = 1_H$. Now you can also show that for a group homomorphism $f : G \to H$, you always have that $f(g^{-1}) = f(g)^{-1}$.

0.2.4. The category $\mathfrak{ab}$ of Abelian groups. This is the full subcategory of the category of groups consisting of all groups G such that $gh = hg$ for all $g, h \in G$. Actually, if a group G is Abelian, we will generally write the operation *additively*, i.e. writing $g + h$ instead of gh to help remind us that $g+h = h+g$. Also, we'll write the identity element of an Abelian group G not as $1 = 1_G$ but as $0 = 0_G$.

0.2.5. The category $\mathfrak{rings}$ of rings. A ring is an Abelian group (written in the additive notation) together with an additional "multiplication" operation written in multiplicative notation satisfying the following axioms:

(R1) $a(b + c) = ab + ac, (b + c)a = ba + ca$ for all $a, b, c \in R$;

(R2) $(ab)c = a(bc)$;

(R3) there exists a distinguished element $1 = 1_R \in R$ such that $1a = a = a1$ for all $a \in R$.

Actually, because I've included (R3), we're always talking about *unital rings*. In other algebra texts like Hungerford, and in other courses especially analysis, you will meet more general, not necessarily unital rings where axiom (R3) is dropped. Note: the identity element $1 \in R$ is uniquely determined (exercise). If R is a ring, we'll always write

$$R^* = \{a \in R \mid a \neq 0\},$$
$$R^\times = \{a \in R \mid \text{there exists an element } a^{-1} \in R \text{ such that } aa^{-1} = 1 = a^{-1}a\}.$$

Elements of $R^\times$ are called *units*.

I haven't defined the category of rings yet, because I haven't defined what a morphism of rings means. This is a morphism $f : R \to S$ of Abelian groups such that $f(ab) = f(a)f(b)$ for all $a, b \in R$, and moreover $f(1_R) = 1_S$. (Some people call this a *unital* homomorphism because of the requirement that 1 goes to 1: this is not in general implied by the multiplicative property).

I should of course mention the ring $\mathbb{Z}$ of integers under usual multiplication and addition. Note for any ring R, there is a unique ring homomorphism $\mathbb{Z} \to R$. Indeed, if $f : \mathbb{Z} \to R$ is a ring homomorphism, then we have no choice but to set $f(0) = 0_R$ and $f(1) = 1_R$. Hence since f is a homomorphism, $f(n) = 1_R + \cdots + 1_R$ (n times) for $n \geq 1$, and $f(-n) = -f(n)$. We'll always write $n.1_R$ for the element $f(n)$. The other basic example of a ring is the *zero ring*, namely the ring with just one element, namely $0 = 1$. In all other rings, we have that $0 \neq 1$...

0.2.6. The category $\mathfrak{fields}$ of fields. A field is a commutative ring, i.e. a ring K in which $ab = ba$ for all $a, b \in K$, such that in addition, $1 \neq 0$ and every non-zero element is a unit. The category of fields is a full subcategory of the category of rings. For example, the ring $\mathbb{Z}_n$ of integers modulo n is a field if and only if n is prime.

0.2.7. The category $\mathfrak{vec}(K)$ of *finite dimensional* vector spaces over a field K. The objects are the finite dimensional vector spaces over K, the morphisms are the K-linear maps between vector spaces.

There are of course many other examples of categories, we'll introduce the ones we need when we need them. You'll also see examples in other subjects, for example the category of topological spaces, morphisms being continuous maps.

0.3 Products and coproducts

The language of category theory is ideally suited for defining general notions which arise again and again when studying examples. In this section, I want to introduce the notions of products and coproducts (a.k.a. direct sums) in a general categorical setting. We'll see them again when studying the category of groups, the category of rings, of modules, etc... so its useful to know there's some motivating theory justifying the terminology in the examples. You should by the way never get carried away by category theory: algebra is really all about *examples* – category theory just provides us with a language to understand the examples in a unified way.

Let me mention first the notions of initial and terminal objects in a category, since these are easy but illustrate the idea of categorical definitions quite well.

An object I in a category $\mathscr{C}$ is called an *initial object* if for every object A in $\mathscr{C}$, there's a unique morphism from I to A. An object T in a category $\mathscr{C}$ is called an *terminal object* if for every object A in $\mathscr{C}$, there's a unique morphism from A to I. An object Z in a category $\mathscr{C}$ is called a *zero object* if its both an initial and a terminal object. Its an exercise to show that if an initial (resp. terminal, resp. zero) object exists (it may not!) then it is unique up to a unique (or *canonical*) isomorphism in $\mathscr{C}$.

For example, in the category of groups, the *trivial group* $\{1\}$ is a zero object. In the category of rings, $\mathbb{Z}$ is an initial object and the zero ring $\{0\}$ is a terminal object – hence, the category of rings has no zero object because these are different. In the category of sets, the empty set is an initial object, and any one element set is a terminal object.

Now we discuss the notion of a *product* in an arbitrary category. So let $\mathscr{C}$ be a category. Suppose that A_i $(i \in I)$ is a family of objects in $\mathscr{C}$. A *product* of the A_i is an object $P \in \mathscr{C}$ together with morphisms $\pi_i : P \to A_i$ for each $i \in I$, such that

(P) given any object Q and morphisms $\phi_i : Q \to A_i$ for each $i \in I$, there exists a unique morphism $\phi : Q \to P$ such that $\pi_i \circ \phi = \phi_i$ for each $i \in I$.

You should convince yourself that a product of an empty set of objects means exactly the same as a terminal object in the category $\mathscr{C}$.

The definition of product is an example of a definition by a *universal property*. It is always the case that if something is defined by a universal property, then (if it exists) it is unique up to a canonical isomorphism. So we generally (abusing notation) say "the" product of the objects A_i and denote it $\prod_{i\in I} A_i$. That is providing of course such an object exists: whether it does depends on the particular category you're working in.

Note I often refer to "products" as "direct products" out of habit... Now for some examples of products in specific categories:

0.3.1. Suppose $\mathscr{C}$ is the category of sets. Given sets A_i $(i \in I)$, their product is the set $\prod_{i\in I} A_i$ (Cartesian product). The maps $\pi_i : \prod_{i\in I} A_i \to A_i$ in the definition of product are just the projections. You can check this really is the product in the category of sets. Note each π_i is in fact surjective (this statement is equivalent to the Axiom of Choice in set theory!!!)

0.3.2. Suppose $\mathscr{C}$ is the category of groups. Given groups G_i $(i \in I)$, their product is the Cartesian product $\prod_{i\in I} G_i$ of the underlying sets, made into a group by defining multiplication "coordinatewise". The projections $\pi_i : \prod_{i\in I} G_i \to G_i$ are group homomorphisms, and this data satisfies the universal property to be the product in the category of groups. Exactly the same construction works in the category of Abelian groups, or in the category of rings (where one needs to define both multiplication and addition coordinatewise).

0.3.3. Finally, suppose $\mathscr{C}$ is the category of fields. Consider the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$ of integers modulo 2 and 3 respectively. What could their product $\mathbb{Z}_2 \times \mathbb{Z}_3$ be? Well, it would have to be a field K which had field homomorphisms to the both of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Hence, since field homomorphisms are always injective, we would have to have that both $1+1 = 0$ and that $1+1+1 = 0$ in K, which implies that $1 = 0$ which is not allowed in a field. Hence, *products do not exist in the category of fields.*

We turn to discussing *coproducts*. Let $\mathscr{C}$ be any category and A_i $(i \in I)$ be a family of objects in $\mathscr{C}$. A *coproduct* of the A_i is an object $C \in \mathscr{C}$ together with morphisms $\iota_i : A_i \to C$ for each $i \in I$, such that

(C) given any object D and morphisms $\phi_i : A_i \to D$ for each $i \in I$, there exists a unique morphism $\phi : C \to D$ such that $\phi \circ \iota_i = \phi_i$ for each $i \in I$.

Observe the definition of coproduct is the definition of product but with all arrows reversed. Moreover, a coproduct of an empty set of objects means exactly the same as an initial object in the category $\mathscr{C}$.

If a coproduct of the A_i exists it is unique up to a canonical isomorphism. So we call it "the" coproduct and denote it by $\coprod_{i \in I} A_i$. Examples:

0.3.4. In the category of sets, the coproduct is just the disjoint union of the sets, the maps ι_i are the inclusions. (This is also the coproduct in the category of topological spaces where you define the open sets in the disjoint union to be disjoint unions of opens sets).

0.3.5. Coproducts in the category of groups exist but are rather nasty things called *free products* and we will not discuss them here. But in the category of *Abelian groups*, coproducts are easy to understand and very important. Let $\{G_i \mid i \in I\}$ be Abelian groups.

Then, their *coproduct* $\coprod_{i \in I} A_i$ is defined to be the set of elements $(a_i)_{i \in I}$ of the Cartesian product $\prod_{i \in I} A_i$ such that $a_i = 0$ for all but finitely many $i \in I$. We usually denote the element $(a_i)_{i \in I}$ of $\coprod_{i \in I} A_i$ as $\sum_{i \in I} a_i$ instead (morally the sum makes sense because we're adding up elements all but finitely many of which are zero!). Given two elements $\sum_{i \in I} a_i$ and $\sum_{i \in I} b_i$ of $\coprod_{i \in I} A_i$, their sum (remember the operation in an Abelian group is written additively!) is simply $\sum_{i \in I}(a_i + b_i)$ – all but finitely many terms are again zero so this makes sense.

Now we define the maps $\iota_i : A_i \to \coprod_{i \in I} A_i$ needed in the definition of coproduct. One has simply that $\iota_i(a) = \sum_{j \in I} a_j$ where $a_j = 0$ for $j \neq i$ and $a_i = a$. Then you need to check that the object we have defined together with these morphisms really is a coproduct in the category of Abelian groups. We will often write

$$\bigoplus_{i \in I} A_i$$

instead of $\coprod_{i \in I} A_i$ and call it the *direct sum* of the Abelian groups A_i instead of the coproduct.

Notice that if the underlying set I is *finite*, then the object $\coprod_{i \in I} A_i$ is exactly the same as the object $\prod_{i \in I} A_i$: that is *finite products and coproducts coincide for Abelian groups.* In particular, if $I = \{1, 2\}$, we get that $A_1 \times A_2$ (the notation for product of just two objects) equals $A_1 \oplus A_2$ (the notation for sum of just two objects). You need to be flexible about which you write!!

0.3.6. Coproducts do not exist (in general) in the category of rings.

0.4 Functors and natural transformations

So categories encompass different sorts of algebraic structures. We should also allow "maps" between categories if we wish to be able to compare different algebraic structures.

Let $\mathscr{A}, \mathscr{B}$ be categories. A functor $F : \mathscr{A} \to \mathscr{B}$ means:

(F1) a rule assigning to each object $A \in \mathscr{A}$ an object $FA \in \mathscr{B}$;

(F2) a rule assigning to each arrow $f : A \to B$ in $\mathscr{A}$ an arrow $Ff : FA \to FB$ in $\mathscr{B}$.

These rules should satisfy the following axioms:

(F3) $F(f \circ g) = Ff \circ Fg$ for all arrows f, g in $\mathscr{A}$ whose composition makes sense;

(F4) $F\,\mathrm{id}_A = \mathrm{id}_{FA}$ for each object A in $\mathscr{A}$.

I'll just give two examples of functors right now. We'll meet many more as we go along. First, let $\mathscr{C}$ be any category. Then, there's a functor $\mathrm{Id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ defined to be the identity on objects and morphisms. It's called the *identity functor*. Second, define a functor $F : \mathfrak{groups} \to \mathfrak{sets}$ by setting $FG = G$ for a group G (i.e. FG is the same underlying set as G but we've just forgotten that its a set with additional group structure). On a morphism $f : G \to H$, let Ff be the same function, viewed just as a map of sets. A functor like this one F is called a *forgetful functor*: we're just calling the group a set and forgetting the extra group structure!

Strictly speaking, people call a *functor* as I've defined it above a *covariant functor*. There's also another notion of *contravariant functor*. Its almost the same thing, but given a morphism $f : A \to B$ in $\mathscr{A}$, a contravariant functor sends it to a morphism $Ff : FB \to FA$ in $\mathscr{B}$ (before it was $Ff : FA \to FB$). The composition axiom (F3) becomes $F(f \circ g) = Fg \circ Ff$ for all morphisms f, g whose composition makes sense. Informally, *contravariant functors reverse the directions of arrows.*

I prefer usually to use the word functor for covariant functor. To eliminate the need for defining everything twice, once for covariant once for contravariant functors, I'll use a trick: the *opposite* of a category. So let $\mathscr{C}$ be a category. Define $\mathscr{C}^{\mathrm{op}}$, the *opposite category*, as follows. The objects are the same. Moreover, given objects A, B, the set of morphisms $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B)$ is defined to be the set $\mathrm{Hom}_{\mathscr{C}}(B, A)$. Given arrows $f : A \to B$ and $g : B \to C$ in $\mathscr{C}^{\mathrm{op}}$ (i.e. arrows $f : B \to A$ and $g : C \to B$ in $\mathscr{C}$) the composite $g \circ f$ in the category $\mathscr{C}^{\mathrm{op}}$ is defined to be the same as the composite $f \circ g$ in the category $\mathscr{C}$. So the category $\mathscr{C}^{\mathrm{op}}$ is really "the same" as the category $\mathscr{C}$ but with the direction of all arrows reversed. With this trick in mind, we then have that a contravariant functor $F : \mathscr{A} \to \mathscr{B}$ is the same as an ordinary (covariant) functor $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$ (or for that matter $F : \mathscr{A} \to \mathscr{B}^{\mathrm{op}}$).

For an example of a contravariant functor, consider the category $\mathfrak{vec}(K)$ of finite dimensional vector spaces over a field K. Given such a vector space V, we have the *dual space*

$$V^* = \mathrm{Hom}_K(V, K)$$

of all linear functionals on V. If $e_1, \dots, e_n$ is a basis for V, then we obtain a basis for V^* by considering the *dual basis* $f_1, \dots, f_n$. Here, f_i is the linear map from V to K determined uniquely by the formula

$$f_i(e_j) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

So V^* is another vector space of the same dimension n as V itself.

Now given a morphism $\theta : V \to W$, i.e. a linear transformation between finite dimensional vector spaces, we can define the *dual map* $\theta^* : W^* \to V^*$ as follows. Take $f \in W^*$. So f is a linear map from W to K. Then, $\theta^* f$ is the element of V^*, i.e. the linear map from V to K, defined by

$$(\theta^* f)(v) = f(\theta(v))$$

for all $v \in V$.

Now we can define a *contravariant functor* $D : \mathfrak{vec}(K) \to \mathfrak{vec}(K)$ by setting $D(V) = V^*$ for an object V and $D(f) = f^*$ for a morphism f. Of course, you need to check things like $D(f \circ g) = D(g) \circ D(f)$, i.e. $(f \circ g)^* = g^* \circ f^*$, to verify that this really is a contravariant functor. Alternatively, you can think of D as an ordinary (covariant) functor from $\mathfrak{vec}(K)^{\mathrm{op}} \to \mathfrak{vec}(K)$ or $\mathfrak{vec}(K) \to \mathfrak{vec}(K)^{\mathrm{op}}$.

Composing the functor D with itself, we obtain a functor $D \circ D : \mathfrak{vec}(K) \to \mathfrak{vec}(K)$. So $(D \circ D)(V) = (V^*)^*$ and $(D \circ D)(f) = (f^*)^*$. Now, for each finite dimensional vector space V, there is a linear map

$$i_V : V \to (V^*)^* \tag{1}$$

defined so that for each $v \in V$, $i_V(v)$ is the functional on V^* such that $i_V(v)(f) = f(v)$ for each $f \in V^*$. You should check that $i_V(e_1), \dots, i_V(e_n)$ is simply the basis for V^* that's dual to the dual basis $f_1, \dots, f_n$ for V^*. In other words, $i_V : V \to (V^*)^*$ is an isomorphism since it maps a basis to a basis.

This example brings us to the next notion in category theory. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. They are called *isomorphic* if there exist (covariant) functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ such that $F \circ G = \mathrm{Id}_{\mathscr{B}}$ and $G \circ F = \mathrm{Id}_{\mathscr{A}}$. The "duality" functor $D : \mathfrak{vec}(K)^{\mathrm{op}} \to \mathfrak{vec}(K)$ defined above really ought to be an isomorphism of categories. The inverse functor $\mathfrak{vec}(K) \to \mathfrak{vec}(K)^{\mathrm{op}}$ really ought to be just D itself again. But *it is not*!!!! Indeed, $D \circ D(V) = (V^*)^*$ which, although *isomorphic to* V is not *equal to* V. This is maybe an indication that the notion of isomorphism of categories is not the useful one. Instead we need something slightly weaker called *equivalence of categories*.

Two categories $\mathscr{A}$ and $\mathscr{B}$ are said to be *equivalent* if there exist functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ such that $F \circ G \cong \mathrm{Id}_{\mathscr{B}}$ and $G \circ F \cong \mathrm{Id}_{\mathscr{A}}$. (Almost the same as the definition of isomorphic categories in the previous paragraph but the functors are only $\cong$ Id not $=$ Id!) We do not understand this definition yet, because I have not defined what it means for functors to be *isomorphic* rather than equal. For this, we need the notion of *natural transformation of functors*.

So finally, let $F, G : \mathscr{A} \to \mathscr{B}$ be two functors. A *natural transformation* $\eta : F \to G$ means the following data: for each object $A \in \mathscr{A}$, a morphism $\eta_A : F(A) \to G(A)$ in the category $\mathscr{B}$. These maps η_A need to satisfy the following axiom: for each arrow $f : A \to B$ in $\mathscr{A}$, we have that $\eta_B \circ Ff = Gf \circ \eta_A$.

A natural transformation is called a *natural isomorphism* between the functors F and G if in addition each morphism η_A is actually an isomorphism in the category $\mathscr{B}$. If F and G are naturally isomorphic functors, we write $F \cong G$.

Now let's go back to our example, the functor $D : \mathfrak{vec}(K)^{\mathrm{op}} \to \mathfrak{vec}(K)$. I claim that D is an equivalence of categories (so that $\mathfrak{vec}(K)$ is equivalent to its opposite category). To prove this, consider the functor $D : \mathfrak{vec}(K) \to \mathfrak{vec}(K)^{\mathrm{op}}$. I claim that $D \circ D \cong \mathrm{Id}$ (either way round). To prove this, we need a natural isomorphism between these two functors. But we wrote down a natural isomorphism $i : \mathrm{Id} \to D \circ D$ of functors earlier (1) (you should check this of course).

We will see more of these notions of natural transformations later on. It is a difficult definition to get your head round! It is good enough at this stage to think of natural transformations rather informally: it simply means that your definition *doesn't depend on any specific information about the object you are considering.*

0.5 *Additive categories*

I want to include here one last categorical notion. This section is starred, which means we did not cover it in class and you shouldn't waste time on it when preparing for exams!!

A category $\mathscr{C}$ is called an *additive category* if

(A1) for all objects $A, B \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ has the additional structure of an Abelian group (the operation written $f + g : A \to B$ for arrows $f, g : A \to B$);

(A2) addition distributes over composition, i.e. $f \circ (g + g') = f \circ g + f \circ g'$ and $(f + f') \circ g = f \circ g + f' \circ g$ for arrows $f, f' : B \to C$ and $g, g' : A \to B$;

(A3) each finite family of objects in $\mathscr{C}$ has both a product and a coproduct.

Note axiom (A3) in particular implies that the empty family consisting of no objects in $\mathscr{C}$ has both a product and a coproduct. This means that $\mathscr{C}$ has both a terminal and an initial object. Actually, it would be enough in axiom (A3) just to require that $\mathscr{C}$ has an initial and a terminal object and that every pair of objects in $\mathscr{C}$ possesses a product and a coproduct.

For finite families of objects in a category $\mathscr{C}$ satisfying axioms (A1) and (A2), there is a third notion of product called a *biproduct* which connects the two notions of product and coproduct discussed so far. Let $X_1, \ldots, X_n$ be objects in $\mathscr{C}$. A *biproduct* of $X_1, \ldots, X_n$, to be an object X

together with maps $p_i : X \to X_i, q_i : X_i \to X$ such that

$$p_j \circ q_i = \delta_{i,j}\,\mathrm{id}_{X_i}, \qquad \sum_{i=1}^{n} q_i \circ p_i = \mathrm{id}_X .$$

Here, $\delta_{i,j}\,\mathrm{id}_{X_i}$ denotes the identity map $X_i \to X_i$ in case $i = j$ and the zero map $X_i \to X_j$ (the zero in the Abelian group $\mathrm{Hom}_{\mathscr{C}}(X_i, X_j)$) in case $i \neq j$.

Theorem. *In a category $\mathscr{C}$ satisfying axioms (A1) and (A2) above, let $X_1, \ldots, X_n, X$ be objects and $p_i : X \to X_i, q_i : X_i \to X$ be maps such that $p_j \circ q_i = \delta_{i,j}\,\mathrm{id}_{X_i}$. Then, the following are equivalent:*
(1) *X together with the maps p_i is a product of the X_i;*
(2) *X together with the maps q_i is a coproduct of the X_i;*
(3) *X together with the maps p_i and q_i is a biproduct of the X_i.*

Proof. We just need to prove (1) and (3) are equivalent. The equivalence of (2) and (3) is then immediate considering the opposite category to $\mathscr{C}$ instead.

(1)⇒(3). Suppose that X together with the p_i is a product. Set $\phi = \sum_{i=1}^{n} q_i \circ p_i$. Then,

$$p_i \circ \phi = \sum_{j=1}^{n} p_i \circ q_j \circ p_j = \sum_{j=1}^{n} \delta_{j,i} p_j = p_i.$$

It follows from the uniqueness in the universal property of products that $\phi = \mathrm{id}_X$, so X is a biproduct.

(3)⇒(1). Suppose we have an object Y and morphisms $f_i : Y \to X_i$ for each i. To show that X is a product, we need to prove there exists a unique $f : Y \to X$ such that $p_i \circ f = f_i$ for each i. Well, $f = \sum_{i=1}^{n} q_i \circ p_i \circ f$ so if $p_i \circ f = f_i$ for each i, then $f = \sum_{i=1}^{n} q_i \circ f_i$. In other words, there is *no choice* but to define

$$f = \sum_{i=1}^{n} q_i \circ f_i.$$

Then,

$$p_j \circ f = \sum_{i=1}^{n} p_j \circ q_i \circ f_i = f_j$$

as required. □

If $\mathscr{C}$ is a category satisfying axioms (A1) and (A2), suppose that X together with maps $p_i : X \to X_i$ is a product of objects $X_1, \ldots, X_n$. Fix j and define $f_i : X_j \to X_i$ for each i by setting $f_i = 0$ if $i \neq j$ and $f_i = \mathrm{id}$ if $i = j$ (recall that 0 makes sense as a morphism by (A1)). By the universal property of products, there exists a unique $q_j : X_j \to X$ such that $p_i \circ q_j = \delta_{i,j}\,\mathrm{id}_{X_i}$. Therefore by the theorem, X together with the maps $q_i : X_i \to X$ is a coproduct of the X_i. Making the same argument in the opposite category too, we obtain:

Corollary. *Let $\mathscr{C}$ be a category satisfying (A1) and (A2). Then any finite family of objects of $\mathscr{C}$ has a product if and only if it has a coproduct, in which case the two are isomorphic.*

In particular, this implies that in an additive category, initial objects and terminal objects are isomorphic, hence there is a *zero object*.

The basic example of an additive category is the category $\mathfrak{ab}$ of Abelian groups. We have already constructed finite products and coproducts (which happened to be isomorphic). The addition of two morphisms $f, g : A_1 \to A_2$ is defined by $(f + g)(a) = f(a) + g(a)$, giving the Abelian group structure on the Hom sets.

Now let $\mathscr{B}$ and $\mathscr{C}$ be two additive categories. A functor $F : \mathscr{B} \to \mathscr{C}$ is called an *additive functor* if $F(f+g) = Ff + Fg$ for all arrows $f, g : X \to Y$ in $\mathscr{B}$. In other words, the set map

$$\mathrm{Hom}_{\mathscr{B}}(X, Y) \to \mathrm{Hom}_{\mathscr{B}}(FX, FY)$$

induced by the functor F is actually a morphism of Abelian groups.

Additive functors commute with finite products. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between additive categories. If X together with maps $p_i : X \to X_i$ is a product of finitely many objects X_i in $\mathscr{A}$, then FX together with the maps $Fp_i : FX \to FX_i$ is a product of the objects FX_i in $\mathscr{B}$.*

Proof. By the theorem (1)⇒(3), there exist maps $q_i : X_i \to X$ such that X together with these maps is a biproduct of the X_i. The result of applying an additive functor to a biproduct is clearly a biproduct. Hence by the theorem (3)⇒(1) it is a product. □

Restating this result for the opposite categories (which are also additive), you obtain the dual statement:

Additive functors commute with finite coproducts. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between additive categories. If X together with maps $q_i : X_i \to X$ is a coproduct of finitely many objects X_i in $\mathscr{A}$, then FX together with the maps $Fq_i : FX_i \to FX$ is a coproduct of the objects FX_i in $\mathscr{B}$.*

These two general nonsense results are very useful. Actually, in the cases we need to apply them (basically tensor and hom functors) it is easy to prove the conclusion directly. But it gives you the flavour of the sort of general facts you can prove by developing category theory.

The point really of introducing an additive category is that notions of image and kernel of a morphism can be defined in an additive category in an abstract way (though they may not exist in general). Then you go on to define something called an *Abelian category*, namely, an additive category with some extra axioms devised exactly so that there is a meaningful notion of *exact sequence* in such a category. For example, $\mathfrak{ab}$ turns out to be an example of an Abelian category (and hence the name!). Fortunately for us, we will be able to avoid such abstraction since we will only consider one specific Abelian category in what follows, namely, the module category of a ring. In this particular category, it is easier to understand the notion of exact sequence without category theory.