EXERCISE SHEET THREE

Exercise 1. In the case $\Gamma = A_n$, we have now given two explicit constructions of the Kac-Moody algebra $\mathfrak{g}(\Gamma)$: as the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$ or as the Lie algebra $\mathfrak{g}(Q)$ for the quiver

$$Q = 1 \to 2 \to \cdots \to n.$$

The goal of this exercise is to identify the two. Let $\mathfrak{h} = \mathbb{C}\epsilon_1 + \cdots + \mathbb{C}\epsilon_n$ as usual. Identify this with the subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$ consisting of the diagonal trace zero matrices, so that $\epsilon_i \leftrightarrow e_{i,i} - e_{i+1,i+1}$ (matrix units). Recall that $\Delta = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$. Prove explicitly that the map

$$h \mapsto h \; (h \in \mathfrak{h}), \quad e_{\epsilon_i - \epsilon_j} \mapsto e_{i,j} \; (i < j), \quad e_{\epsilon_i - \epsilon_j} \mapsto -e_{i,j} \; (i > j)$$

is an isomorphism between $\mathfrak{g}(Q) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}e_\alpha$ and $\mathfrak{sl}_{n+1}(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}e_{i,j}$.

Exercise 2 (Triality)**.** Consider the quiver $Q = D_4$ consisting of three vertices $1, 2, 3$ around the edge, one vertex 4 in the middle with all arrows pointing inwards. Recall that the positive roots are $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_1 + \epsilon_4, \epsilon_2 + \epsilon_4, \epsilon_3 + \epsilon_4, \epsilon_1 + \epsilon_2 + \epsilon_4, \epsilon_2 + \epsilon_3 + \epsilon_4, \epsilon_1 + \epsilon_3 + \epsilon_4, \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4, \epsilon_1 + \epsilon_2 + \epsilon_3 + 2\epsilon_4$. Let ν be the asymmetry function as defined above, and $\mathfrak{g} = \mathfrak{h} + \bigoplus_{\alpha \in \Delta} \mathbb{C}e_\alpha$ be the Lie algebra $\mathfrak{g}(Q)$ ($\cong \mathfrak{so}_8(\mathbb{C})$) as defined above. Let $\tau : \mathfrak{h} \to \mathfrak{h}$ be the automorphism sending $\epsilon_1 \to \epsilon_2 \to \epsilon_3 \to \epsilon_1$ and fixing ϵ_4. (This corresponds to the natural symmetry of the quiver Q so it really is an isometry).

- (a) Prove that $\nu(\alpha, \beta) = \nu(\tau(\alpha), \tau(\beta))$.
- (b) Deduce that setting $\tau(e_\alpha) = e_{\tau(\alpha)}$ for each $\alpha \in \Delta$ defines an automorphism of $\mathfrak{g}$ of order 3.
- (c) Construct this automorphism of $\mathfrak{g}$ in another way using our original definition of $\mathfrak{g}$ as the quotient of $\tilde{\mathfrak{g}}$ by the ideal $\mathfrak{r}$.

Exercise 3. Let $\mathscr{L} = \mathbb{C}[t, t^{-1}]$.

- (i) For $j \in \mathbb{Z}$, show that $d_j = -t^{j+1}\frac{d}{dt}$, i.e. the function $p \mapsto -t^{j+1}\frac{dp}{dt}$, is a derivation of $\mathscr{L}$.
- (ii) Show that $\mathfrak{d} = \operatorname{Der} \mathscr{L} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}d_j$.
- (iii) Show that $\psi : \mathfrak{d} \times \mathfrak{d} \to \mathbb{C}$, $\psi(d_i, d_j) = \frac{1}{12}(i^3 - i)\delta_{i,-j}$ is a 2-cocycle on $\mathfrak{d}$.

The corresponding one dimensional central of d by a 1 dimensional center $\mathbb{C}\delta$ is called the *Virasoro algebra.* It is an important infinite dimensional Lie algebra beloved by mathematical physicists!

Exercise 4. Recall the Chevalley antiautomorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ swapping e_i, f_i's and equal to the identity on $\mathfrak{h}$. What does ω do to δ, d and $t^m \otimes x$ for $x \in \mathring{\mathfrak{g}}$?

Exercise 5. Suppose that Γ is the Dynkin diagram A_{n-1}. We know in this case that the Weyl group W is the symmetric group S_n on generators $s_1, \dots, s_{n-1}$ where $s_i = (i, i{+}1)$ and the Euclidean space E is the subspace of the standard Euclidean space $\mathbb{R}^n = \mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_n$ spanned by $\epsilon_i = v_i - v_{i+1}$ for $i = 1, \dots, n-1$. Use Lemma 3.24 above to show that for $w \in S_n$,

$$\ell(w)$$

is the number of pairs (i, j) with $1 \leq i < j \leq n$ and $w(i) > w(j)$.

Exercise 6. Continue with the above example of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Compute the character of V^*, the dual of the natural module. Hence show that $V^* \cong \bigwedge^{n-1} V$ and prove that the tensor product $V \otimes V^*$ has exactly two composition factors. What are their highest weights?

Exercise 7. Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form $(.,.)$. Let x_i and y_i be a pair of dual basis. Let

$$\Omega_2 = \sum_{i=1}^{n} x_i \otimes y_i.$$

Show that for any $\mathfrak{g}$-module V, the action of the operator Ω_2 on $V \otimes V$ commutes with the usual tensor product action of $\mathfrak{g}$.

Exercise 8. Suppose $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Let $\lambda \in P^+$ be a weight with $(\lambda, \epsilon_1) = a, (\lambda, \epsilon_2) = b$. Prove that

$$\dim L(\lambda) = \frac{1}{2}(a+1)(b+1)(a+b+2).$$

What well known representation is the one with $a = b = 1$?

Exercise 9. Go through the details of the proof that the module $S(\mathfrak{t}_-)$ is an irreducible $\mathfrak{t}$-module.

Exercise 10. Prove that the elements $x_1, \dots, x_n \in \mathbb{Z}S_n$ defined from

$$x_i = \sum_{1 \leq j < i} (i\, j)$$

commute.