EXERCISE SHEET TWO

Exercise 1. Let M be the variety of all $n \times n$ matrices and let $GL_n(k)$ act on M by conjugation. Prove that if $m \in M$ is a semisimple (a.k.a. diagonalizable) matrix, then the conjugacy class of m is closed.

Exercise 2. Suppose Q is the quiver with two vertices and one arrow from 1 to 2. The orbits in $\mathrm{Rep}(\alpha_1, \alpha_2)$ are the equivalence classes of $\alpha_2 \times \alpha_1$ matrices, thus parametrized by their rank $0 \leq r \leq \min(\alpha_1, \alpha_2)$. Compute the dimensions of each of these orbits. Describe the partial order on the orbits defined by $O_r \leq O_s$ if $O_r \subseteq \overline{O_s}$. The closures of these orbits are called rank varieties. These and their analogs for other type A quivers are an important source of tractable examples in algebraic geometry.

Exercise 3. Suppose Q is the quiver considered in Exercise 8 of the previous problem set (one vertex in the middle, 3 round the edge, arrows pointing inwards). You should already calculated the dimensions of the twelve different indecomposables. Let $\alpha = (1, 1, 1, 2)$ (where the 2 is on the inside vertex). Question: how many orbits does $GL(\alpha)$ have on $\mathrm{Rep}(\alpha)$ in this case? Try to describe the partial order on the orbits given by containment of closures in this case.

Exercise 4. For each dimension $r \geq 1$, construct a $k[x], k[t]$-bimodule M_r that is finitely generated and free over $k[t]$ such that any indecomposable $k[x]$-module of dimension r is isomorphic to $M_r \otimes_{k[t]} k[t]/(t - \lambda)$ for $\lambda \in k$. Deduce that the polynomial algebra $k[x]$ is of tame representation type.

Exercise 5. Suppose Γ is Dynkin or Euclidean and $\Gamma \neq \tilde{A}_0, \tilde{A}_1$. Show that all the simple roots are conjugate under W, hence by the lemma the real roots form a single W-orbit. How many orbits of real roots are there in type $\tilde{A}_1$?

Exercise 6. Check the definitions to convince yourself that: for Γ of type D_n, the lattice R can be realized as the lattice inside a Euclidean space with orthonormal basis $v_1, \ldots, v_n$ generated as the $\mathbb{Z}$-span of the vectors $\epsilon_1 = v_1 - v_2, \epsilon_2 = v_2 - v_3, \ldots, \epsilon_{n-1} = v_{n-1} - v_n, \epsilon_n = v_{n-1} + v_n$. Compute the action of the generators $s_1, \ldots, s_n$ of W on R, and hence prove that $\Delta^+ = \{v_i \pm v_j \mid 1 \leq i < j \leq n\}$.

Exercise 7. The purpose of this exercise is to construct the free Lie algebra generated by a vector space V (i.e. the vector space with basis given by the generators you have in mind).

(a) Write down the definition of the free Lie algebra $F(V)$ on the vector space V by universal property.

(b) Here is a construction of $F(V)$. Let $T(V)$ be the tensor algebra on the vector space V, so $T(V)$ is universal amongst all associative algebras generated by the vector space V. Viewing $T(V)$ instead as a Lie algebra, let $F(V)$ be the Lie subalgebra of $T(V)$ generated by the subspace V, i.e. $F(V)$ is the intersection of all the Lie subalgebras of $T(V)$ containing V. *Prove* that $F(V)$ together with the inclusion $V \hookrightarrow F(V)$ IS the free Lie algebra on vector space V.

(c) Now prove that the universal enveloping algebra $U(F(V))$ is isomorphic to the tensor algebra $T(V)$.

Exercise 8. If $\mathfrak{g}$ is any Lie algebra and V, W are $\mathfrak{g}$-modules, there is a natural way to make the tensor product $V \otimes W$ into a $\mathfrak{g}$-module: $x(v \otimes w) := (xv) \otimes w + v \otimes (xw)$.

(a) Let V be a $\mathfrak{g}$-module. Recall the symmetric algebra $S(V)$ is the quotient of $T(V)$ by the ideal generated by $\{x \otimes y - y \otimes x \mid x, y \in V\}$. Verify that this ideal is invariant under the action of $\mathfrak{g}$, hence $S(V)$ is a $\mathfrak{g}$-module. (So is $\bigwedge(V)$).

(b) In the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, let V be the natural 2 dimensional module on standard basis v_1, v_2. Prove that $S^n(V) \cong L(n)$, the irreducible module of dimension $(n+1)$.

Exercise 9. If $\mathfrak{g}$ is any Lie algebra and V is a finite dimensional $\mathfrak{g}$-module, there is a natural way to make the dual space V^* into a $\mathfrak{g}$-module: $(xf)(v) = -f(xv)$ for $f \in V^*, x \in \mathfrak{g}$ and $v \in V$. There is also always a trivial $\mathfrak{g}$-module, namely the one dimensional vector space $\mathbb{C}$ on which each $x \in \mathfrak{g}$ acts as zero.

(a) Suppose that V is a finite dimensional $\mathfrak{g}$-module. Prove that $V \cong V^*$ as $\mathfrak{g}$-modules if and only if there is a non-degenerate bilinear form $(.,.)$ on V which is *invariant* in the sense that $(xv, w) + (v, xw) = 0$ for all $v, w \in V$ and $x \in \mathfrak{g}$.

(b) Suppose $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Prove that $L(n)^* \cong L(n)$ as $\mathfrak{g}$-modules.

(c) In particular consider the case that $V = L(2)$ is the natural two dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Write down explicitly a non-degenerate invariant bilinear form on V, hence deduce that $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C})$.

(d) Do the same if $V = L(3)$ and hence show that $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C})$.

Exercise 10. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, on standard basis e, h, f with relations $[e, f] = h, [h, e] = 2e, [h, f] = -2f$.

(a) Prove that the element $c := fe + \frac{1}{4}h(h+2)$ belongs to the center of the universal enveloping algebra $U(\mathfrak{g})$. (Hint: you need to show it commutes with each of the generators e, h, f of $U(\mathfrak{g})$. It is useful to note that $[x, yz] = [xy]z + y[xz]$ for $x \in \mathfrak{g}$ and $y, z \in U(\mathfrak{g})$, i.e. $\operatorname{ad} x$ acts on $U(\mathfrak{g})$ as a derivation.)

(b) Show that c acts on the irreducible module $L(n)$ as the scalar $\frac{1}{4}n(n+2)$. Deduce that any short exact sequence

$$0 \longrightarrow L(n) \longrightarrow V \longrightarrow L(m) \longrightarrow 0$$

of $\mathfrak{g}$-modules splits for $m \neq n$.

(c) Prove that the short exact sequence also splits in the case $m = n$ (hint: think about the h-weight space of eigenvalue n first).
(d) Deduce that any finite dimensional $\mathfrak{g}$-module is completely reducible.