

EXERCISE SHEET ONE

Exercise 1. Let M be an R -module and let $e \in R$ be an idempotent (i.e. $e^2 = e$). Then, eRe is a subring of R (but it is not a unital subring). Show that

$$eM \cong \text{Hom}_R(Re, M)$$

as eRe -modules and that

$$eRe \cong \text{End}_R(Re)^{\text{op}}$$

as rings.

Exercise 2. (i) Prove that $\text{rad}(M \oplus N) = \text{rad } M \oplus \text{rad } N$.

(ii) Prove directly that the algebra $M_n(D)$ of $n \times n$ matrices over a division ring is a simple ring, i.e. it has no non-trivial two-sided ideals.

(iii) Deduce from (i) and (ii) that a finite direct sum of matrix algebras over division rings is a semisimple ring.

Exercise 3. Let M be an R -module, and let X, Y be submodules such that M/X is semisimple and M/Y is irreducible. Prove that $M/(X \cap Y)$ is semisimple. Hence prove that if M satisfies DCC, then M is semisimple if and only if $\text{rad } M = 0$.

Exercise 4. Classify the indecomposable modules of the group algebra kC_n of the cyclic group of order n over an algebraically closed field of characteristic p .

Exercise 5. Let $A = T_n(k)$ be the algebra of all upper triangular $n \times n$ matrices over a field k . What is $J(A)$? How many inequivalent irreducible A -modules are there? What are their dimensions? What do their projective covers look like?

Exercise 6. Suppose Q is a quiver with at most one path between any two points. Then, kQ is isomorphic to the subalgebra of $M_n(k)$ consisting of all matrices with ij -entry equal to zero if there is no path from j to i . For example,

$$1 \rightarrow 2 \rightarrow \dots \rightarrow n$$

is the lower triangular matrices.

Exercise 7. (*Some indecomposable representations of the Kronecker quiver*) Recalling Example 2.11(3), let Q be the Kronecker quiver. Fix $\lambda \in k$ and $n \geq 1$. Take the representation $V = V(\lambda, n)$ defined by letting $V_1 = V_2 = k^n$ and the linear maps on the two arrows $V_1 \rightarrow V_2$ being the maps $I_n : k^n \rightarrow k^n$ (the identity matrix) and $J_n(\lambda) : k^n \rightarrow k^n$ (the Jordan block of size n with eigenvalue λ). Show that $V(\lambda, n)$'s are (infinitely many) inequivalent indecomposable representations of Q . Up to isomorphism, there is exactly

one other indecomposable representation in which the vector spaces V_1 and V_2 have dimension n . What is it?

Exercise 8. (*three subspace problem*) Find 12 non-isomorphic indecomposable representations (not counting the zero representation as an indecomposable) of the quiver with one vertex in the middle and three vertices around the edge, with three arrows all pointing inwards. The general theory developed in a while will show these are *all* the indecomposables, so you don't need to prove that extra thing right now.

Exercise 9. (*four subspace problem*) Find infinitely many non-isomorphic indecomposable representations of the quiver with one vertex in the middle and four vertices around the edge, with four arrows all pointing inwards.

Exercise 10. Take the two quivers discussed in Example 2.13(5) (both had underlying graph $1 - 2 - 3$). For each, write down the matrix of the asymmetric bilinear form $\langle \cdot, \cdot \rangle$ and of the symmetric bilinear form (\cdot, \cdot) with respect to the basis $\epsilon_1, \epsilon_2, \epsilon_3$ for \mathbb{Z}^3 . This is the Cartan matrix of type A_3 !