

4. THE REPRESENTATION THEORY OF THE SYMMETRIC GROUP

In this chapter we give an overview of the p -modular representation theory of the symmetric group S_n and its connection to the affine Kac-Moody algebra of type $A_{p-1}^{(1)}$. We work over a field F of arbitrary characteristic p .

4.1. Formal characters. For $k = 1, \dots, n$, we define the *Jucys-Murphy element*

$$x_k := \sum_{i=1}^{k-1} (i \ k) \in FS_n, \quad (1)$$

see [15, 28]. It is straightforward to show that the elements x_1, x_2, \dots, x_n commute with one another. Moreover, we have by [15] or [29, 1.9]:

Theorem 4.1. *The center of the group algebra FS_n is precisely the set of all symmetric polynomials in the elements x_1, x_2, \dots, x_n .*

Now let M be an FS_n -module. Let $I = \mathbb{Z}/p\mathbb{Z}$ identified with the prime subfield of F . For $\underline{i} = (i_1, \dots, i_n) \in I^n$, define

$$M[\underline{i}] := \{v \in M \mid (x_r - i_r)^N v = 0 \text{ for } N \gg 0 \text{ and each } r = 1, \dots, n\}.$$

Thus, $M[\underline{i}]$ is the simultaneous generalized eigenspace for the commuting operators x_1, \dots, x_n corresponding to the eigenvalues i_1, \dots, i_n respectively.

Lemma 4.2. *Any FS_n -module M decomposes as $M = \bigoplus_{\underline{i} \in I^n} M[\underline{i}]$.*

Proof. It suffices to show that all eigenvalues of x_r on M lie in I , for each $r = 1, \dots, n$. This is obvious if $r = 1$ (as $x_1 = 0$). Now assume that all eigenvalues of x_r on M lie in I , and consider x_{r+1} . Let $v \in M$ be a simultaneous eigenvector for the commuting operators x_r and x_{r+1} . Consider the subspace N spanned by v and $s_r v$.

Suppose that N is two dimensional. Then the matrix for the action of x_r on N with respect to the basis $\{v, s_r v\}$ is $\begin{pmatrix} i & c \\ 0 & j \end{pmatrix}$ for some $i, j \in I$ and $c \in F$ (by assumption on the eigenvalues of x_r). Hence, the matrix for the action of $x_{r+1} = s_r x_r s_r + s_r$ on N is $\begin{pmatrix} j & 1 \\ c+1 & i \end{pmatrix}$. Since v was an eigenvector for x_{r+1} , we see that $c = -1$, hence v has eigenvalue j for x_{r+1} as required.

Finally suppose that N is one dimensional. Then, $s_r v = \pm v$. Hence, if $x_r v = i v$ for $i \in I$, then $x_{r+1} v = (s_r x_r s_r + s_r) v = (i \pm 1) v$. Since $i \pm 1 \in I$, we are done. \square

We define the *formal character* $\text{ch } M$ of a finite dimensional FS_n -module M to be

$$\text{ch } M := \sum_{\underline{i} \in I^n} \dim(M[\underline{i}]) e^{\underline{i}}, \quad (2)$$

an element of the free \mathbb{Z} -module on basis $\{e^{\underline{i}} \mid \underline{i} \in I^n\}$. This is a useful notion, since ch is clearly additive on short exact sequences and we have the following important result proved in [38, §5.5]:

Theorem 4.3. *The formal characters of the inequivalent irreducible FS_n -modules are linearly independent.*

Given $\underline{i} = (i_1, \dots, i_n) \in I^n$, define its *weight* $\text{wt}(\underline{i})$ to be the tuple $\gamma = (\gamma_i)_{i \in I}$ where γ_j counts the number of i_r ($r = 1, \dots, n$) that equal j . Thus, γ is an element of the set Γ_n of I -tuples of non-negative integers summing to n . Clearly $\underline{i}, \underline{j} \in I^n$ lie in the same S_n -orbit (under the obvious action by place permutation) if and only if $\text{wt}(\underline{i}) = \text{wt}(\underline{j})$, hence Γ_n parametrizes the S_n -orbits on I^n .

For $\gamma \in \Gamma_n$ and an FS_n -module M , we let

$$M[\gamma] := \sum_{\underline{i} \in I^n \text{ with } \text{wt}(\underline{i}) = \gamma} M[\underline{i}]. \quad (3)$$

Unlike the $M[\underline{i}]$, the subspaces $M[\gamma]$ are actually FS_n -submodules of M . Indeed, as an elementary consequence of Theorem 4.1 and Lemma 4.2, we have:

Lemma 4.4. *The decomposition $M = \bigoplus_{\gamma \in \Gamma_n} M[\gamma]$ is precisely the decomposition of M into blocks as an FS_n -module.*

We will say that an FS_n -module M *belongs to the block* γ if $M = M[\gamma]$.

4.2. Induction and restriction operators. Now that we have the notion of formal character, we can introduce the *i -restriction* and *i -induction operators* e_i and f_i . Suppose that $\gamma \in \Gamma_n$. Let $\gamma + i \in \Gamma_{n+1}$ be the tuple $(\delta_i)_{i \in I}$ with $\delta_j = \gamma_j$ for $j \neq i$ and $\delta_i = \gamma_i + 1$. Similarly, assuming this time that $\gamma_i > 0$, let $\gamma - i \in \Gamma_{n-1}$ be the tuple $(\delta_i)_{i \in I}$ with $\delta_j = \gamma_j$ for $j \neq i$ and $\delta_i = \gamma_i - 1$.

If M is an FS_n -module belonging to the block $\gamma \in \Gamma_n$, define

$$e_i M := (\text{res}_{S_{n-1}}^{S_n} M)[\gamma - i] \quad (\text{interpreted as } 0 \text{ in case } \gamma_i = 0), \quad (4)$$

$$f_i M := (\text{ind}_{S_n}^{S_{n+1}} M)[\gamma + i]. \quad (5)$$

Extending additively to arbitrary FS_n -modules M using Lemma 4.4 and making the obvious definition on morphisms, we obtain exact functors

$$e_i : FS_n\text{-mod} \rightarrow FS_{n-1}\text{-mod} \quad \text{and} \quad f_i : FS_n\text{-mod} \rightarrow FS_{n+1}\text{-mod}.$$

The definition implies:

Lemma 4.5. *For an FS_n -module M we have*

$$\text{res}_{S_{n-1}}^{S_n} M \cong \bigoplus_{i \in I} e_i M, \quad \text{ind}_{S_n}^{S_{n+1}} M \cong \bigoplus_{i \in I} f_i M.$$

Note that $e_i M$ can be described alternatively as the generalized eigenspace of x_n acting on M corresponding to the eigenvalue i . This means that the

effect of e_i on characters is easy to describe:

$$\text{if } \text{ch } M = \sum_{\mathbf{i} \in I^n} a_{\mathbf{i}} e^{\mathbf{i}} \quad \text{then} \quad \text{ch}(e_i M) = \sum_{\mathbf{i} \in I^{n-1}} a_{(i_1, \dots, i_{n-1}, i)} e^{\mathbf{i}}. \quad (6)$$

Let us also mention that there are higher *divided power functors* $e_i^{(r)}, f_i^{(r)}$ for each $r \geq 1$. To define them, start with an FS_n -module M belonging to the block γ . Let $\gamma + i^r = \gamma + i + i + \dots + i$ (r times), and define $\gamma - i^r$ similarly (assuming $\gamma_i \geq r$). View M instead as an $F(S_n \times S_r)$ -module by letting S_r act trivially. Embedding $S_n \times S_r$ into S_{n+r} in the obvious way, we then define

$$f_i^{(r)} M := (\text{ind}_{S_n \times S_r}^{S_{n+r}} M)[\gamma + i^r]. \quad (7)$$

Extending additively, we obtain the functor $f_i^{(r)} : FS_n\text{-mod} \rightarrow FS_{n+r}\text{-mod}$. This exact functor has a two-sided adjoint $e_i^{(r)} : FS_{n+r}\text{-mod} \rightarrow FS_n\text{-mod}$. It is defined on a module M belonging to block γ by

$$e_i^{(r)} M := (M^{S_r})[\gamma - i^r] \quad (\text{interpreted as zero if } \gamma_i < r), \quad (8)$$

where M^{S_r} denotes the space of fixed points for the subgroup $S_r < S_{n+r}$ that permutes $n+1, \dots, n+r$, viewed as a module over the subgroup $S_n < S_{n+r}$ that permutes $1, \dots, n$. The following lemma relates the divided power functors $e_i^{(r)}$ and $f_i^{(r)}$ to the original functors e_i, f_i :

Lemma 4.6. *For an FS_n -module M we have*

$$e_i^r M \cong (e_i^{(r)} M)^{\oplus r!}, \quad f_i^r M \cong (f_i^{(r)} M)^{\oplus r!}.$$

The functors $e_i^{(r)}$ and $f_i^{(r)}$ have been defined in an entirely different way by Grojnowski [9, §8.1], which is the key to proving their properties including Lemma 4.6.

4.3. The affine Kac-Moody algebra. Let R_n denote the character ring of FS_n , i.e. the free \mathbb{Z} -module spanned by the formal characters of the irreducible FS_n -modules. In view of Theorem 4.3, the map ch induces an isomorphism between R_n and the Grothendieck group of the category of all finite dimensional FS_n -modules. Similarly, let R_n^* denote the \mathbb{Z} -submodule of R_n spanned by the formal characters of the projective indecomposable FS_n -modules. This time, the map ch induces an isomorphism between R_n^* and the Grothendieck group of the category of all finite dimensional projective FS_n -modules.

Let

$$R = \bigoplus_{n \geq 0} R_n, \quad R^* = \bigoplus_{n \geq 0} R_n^* \subseteq R. \quad (9)$$

The exact functors e_i and f_i induce \mathbb{Z} -linear operators on R . Since induction and restriction send projective modules to projective modules, Lemma 4.5 implies that e_i and f_i do too. Hence, $R^* \subseteq R$ is invariant under the action of e_i and f_i .

Extending scalars we get \mathbb{C} -linear operators e_i and f_i on $R_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{Z}} R = \mathbb{C} \otimes_{\mathbb{Z}} R^*$. There is also a non-degenerate symmetric bilinear form on $R_{\mathbb{C}}$, the usual *Cartan pairing*, with respect to which the characters of the projective indecomposables and the irreducibles form a pair of dual bases.

Theorem 4.7. *The operators e_i and f_i ($i \in I$) on $R_{\mathbb{C}}$ satisfy the defining relations of the Chevalley generators of the affine Kac-Moody Lie algebra \mathfrak{g} of type $A_{p-1}^{(1)}$ (resp. A_{∞} in case $p = 0$), see [16]. Moreover, viewing $R_{\mathbb{C}}$ as a \mathfrak{g} -module in this way,*

- (i) $R_{\mathbb{C}}$ is isomorphic to the basic representation $V(\Lambda_0)$ of \mathfrak{g} , generated by the highest weight vector e^0 (the character of the irreducible FS_0 -module);
- (ii) the decomposition of $R_{\mathbb{C}}$ into blocks coincides with its weight space decomposition with respect to the standard Cartan subalgebra of \mathfrak{g} ;
- (iii) the Cartan pairing on $R_{\mathbb{C}}$ coincides with the Shapovalov form satisfying $(e^0, e^0) = 1$;
- (iv) the lattice $R^* \subset R_{\mathbb{C}}$ is the \mathbb{Z} -submodule of $R_{\mathbb{C}}$ generated by e^0 under the action of the operators $f_i^{(r)} = f_i^r / r!$ ($i \in I, r \geq 0$);
- (v) the lattice $R \subset R_{\mathbb{C}}$ is the dual lattice to R^* under the Shapovalov form.

This was essentially proved by Lascoux-Leclerc-Thibon [21] and Ariki [1] (for a somewhat different situation), and another approach has been given more recently by Grojnowski [9, 14.2],[10].

4.4. The crystal graph. In view of Theorem 4.7, we can identify $R_{\mathbb{C}}$ with the basic representation of the affine Kac-Moody algebra $\mathfrak{g} = A_{p-1}^{(1)}$. Associated to this highest weight module, Kashiwara has defined a purely combinatorial object known as a *crystal*, see e.g. [18] for a survey of this amazing theory. We now review the explicit description of this particular crystal, due originally to Misra and Miwa [26]. This contains all the combinatorial notions we need to complete our exposition of the representation theory.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ be a partition. We identify λ with its *Young diagram*

$$\lambda = \{(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda_r\}.$$

Elements $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ are called *nodes*. We label each node $A = (r, s)$ of λ with its *residue* $\text{res } A \in I$ defined so that $\text{res } A \equiv (s - r) \pmod{p}$, see Example 4.8 below.

Let $i \in I$ be some fixed residue. A node $A \in \lambda$ is called *i -removable* (for λ) if

(R0) $\text{res } A = i$ and $\lambda - \{A\}$ is the diagram of a partition.

Similarly, a node $B \notin \lambda$ is called *i -addable* (for λ) if

(A0) $\text{res } B = i$ and $\lambda \cup \{B\}$ is the diagram of a partition.

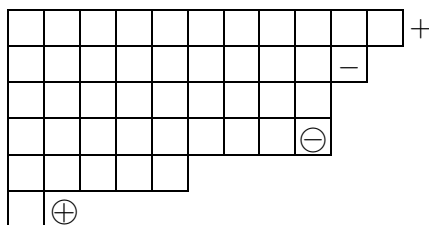
Now label all i -addable nodes of the diagram λ by $+$ and all i -removable nodes by $-$. The *i -signature* of λ is the sequence of pluses and minuses obtained by going along the rim of the Young diagram from bottom left

to top right and reading off all the signs. The *reduced i -signature* of λ is obtained from the i -signature by successively erasing all neighbouring pairs of the form $-+$.

Example 4.8. Let $p = 3$ and $\lambda = (11, 10, 9, 9, 5, 1)$. The residues are as follows:

0	1	2	0	1	2	0	1	2	0	1
2	0	1	2	0	1	2	0	1	2	
1	2	0	1	2	0	1	2	0		
0	1	2	0	1	2	0	1	2		
2	0	1	2	0						
1										

The 2-addable and 2-removable nodes are as labelled in the diagram:



Hence, the 2-signature of λ is $+, -, -, +$ and the reduced 2-signature is $+, -$ (the nodes corresponding to the reduced 2-signature have been circled in the above diagram).

Note the reduced i -signature always looks like a sequence of $+$'s followed by $-$'s. Nodes corresponding to a $-$ in the reduced i -signature are called *i -normal*, nodes corresponding to a $+$ are called *i -conormal*. The leftmost i -normal node (corresponding to the leftmost $-$ in the reduced i -signature) is called *i -good*, and the rightmost i -conormal node (corresponding to the rightmost $+$ in the reduced i -signature) is called *i -cogood*.

We recall finally that a partition λ is called *p -regular* if it does not have p non-zero equal parts. It is important to note that if λ is p -regular and A is an i -good node, then $\lambda - \{A\}$ is also p -regular. Similarly if B is an i -cogood node, then $\lambda \cup \{B\}$ is p -regular.

By [26], the *crystal graph* associated to the basic representation $V(\Lambda_0)$ of \mathfrak{g} can now be realized as the set of all p -regular partitions, with a directed edge $\lambda \xrightarrow{i} \mu$ of color $i \in I$ if μ is obtained from λ by adding an i -cogood node (equivalently, λ is obtained from μ by removing an i -good node). An example showing part of the crystal graph for $p = 2$ is listed below.

4.5. The modular branching graph. Now we explain the relationship between the crystal graph and representation theory. The next lemma was first proved in [20], and in a different way in [11].

Theorem 4.10 has some important consequences. To start with, it implies that the isomorphism classes of irreducible FS_n -modules are parametrized by the vertices in the crystal graph, i.e. by p -regular partitions. For a p -regular partition λ of n , we let D^λ denote the corresponding irreducible FS_n -module. To be quite explicit about this labelling, choose a path

$$\emptyset \xrightarrow{i_1} \square \xrightarrow{i_2} \dots \xrightarrow{i_n} \lambda$$

in the crystal graph from the empty partition to λ , for $i_1, \dots, i_n \in I$. Then,

$$D^\lambda := \tilde{f}_{i_n} \dots \tilde{f}_{i_1} D^\emptyset, \quad (11)$$

where D^\emptyset denotes the irreducible FS_0 -module. Note the labelling of the irreducible module D^λ defined here is known to agree with the standard labelling of James [13], although James' construction is quite different.

Let us state one more result about the structure of the modules $e_i D^\lambda$ and $f_i D^\lambda$, see [2, Theorems E, E'] for this and some other more detailed results.

Theorem 4.11. *Let λ be a p -regular partition of n .*

- (i) *Suppose that A is an i -removable node such that $\mu := \lambda - \{A\}$ is p -regular. Then, $[e_i D^\lambda : D^\mu]$ is the number of i -normal nodes to the right of A (counting A itself), or 0 if A is not i -normal.*
- (ii) *Suppose that B is an i -addable node such that $\nu := \lambda \cup \{B\}$ is p -regular. Then, $[f_i D^\lambda : D^\nu]$ is the number of i -conormal nodes to the left of B (counting B itself), or 0 if B is not i -conormal.*

4.6. More on characters. Let M be an FS_n -module. Define

$$\varepsilon_i(M) = \max\{r \geq 0 \mid e_i^r M \neq 0\} \quad \varphi_i(M) = \max\{r \geq 0 \mid f_i^r M \neq 0\}. \quad (12)$$

Note $\varepsilon_i(M)$ can be computed just from knowledge of the character of M : it is the maximal r such that $e^{(\dots, i^r)}$ appears with non-zero coefficient in $\text{ch } M$. Less obviously, $\varphi_i(M)$ can also be read off from the character of M . By additivity of f_i , we may assume that M belongs to the block $\gamma \in \Gamma_n$. Then

$$\varphi_i(M) = \varepsilon_i(M) + \delta_{i,0} - 2\gamma_i + \gamma_{i-1} + \gamma_{i+1}, \quad (13)$$

see [9, 12.6]. We note the following extremely useful lemma from [11], see also [9, §9]:

Lemma 4.12. *Let D be an irreducible FS_n -module, $\varepsilon = \varepsilon_i(D)$, $\varphi = \varphi_i(D)$. Then, $e_i^{(\varepsilon)} D \cong \tilde{e}_i^\varepsilon D$, $f_i^{(\varphi)} D \cong \tilde{f}_i^\varphi D$.*

The lemma implies that

$$\varepsilon_i(D) = \max\{r \geq 0 \mid \tilde{e}_i^r D \neq 0\}, \quad \varphi_i(D) = \max\{r \geq 0 \mid \tilde{f}_i^r D \neq 0\}.$$

Thus, $\varepsilon_i(D)$ can also be read off directly from the combinatorics: if $D \cong D^\lambda$, then $\varepsilon_i(D)$ is the number of $'-'$'s in the reduced i -signature of λ . Similarly, $\varphi_i(D)$ is the number of $'+'$'s in the reduced i -signature of λ .

Now we can describe an inductive algorithm to determine the label of an irreducible FS_n -module D purely from knowledge of its character $\text{ch } D$. Pick

$i \in I$ such that $\varepsilon := \varepsilon_i(D)$ is non-zero. Let $E = e_i^{(\varepsilon)}D$, an irreducible $FS_{n-\varepsilon}$ -module with explicitly known character thanks to Lemmas 4.12, 4.6 and (6). By induction, the label of E can be computed purely from knowledge of its character, say $E \cong D^\lambda$. Then, $D \cong \tilde{f}_i^\varepsilon E \cong D^\mu$ where μ is obtained from λ by adding the rightmost ε of the i -conormal nodes.

We would of course like to be able to reverse this process: given a p -regular partition λ of n , we would like to be able to compute the character of the irreducible FS_n -module D^λ . One can compute a quite effective *lower bound* for this character inductively using the branching rules of Theorem 4.11. But only over \mathbb{C} is this lower bound always correct: indeed if $p = 0$ then D^λ is equal to the Specht module S^λ and

$$\text{ch } S^\lambda = \sum_{(i_1, \dots, i_n)} e^{(i_1, \dots, i_n)} \quad (14)$$

summing over all paths $\emptyset \xrightarrow{i_1} \square \xrightarrow{i_2} \dots \xrightarrow{i_n} \lambda$ in the characteristic zero crystal graph (a.k.a. Young's partition lattice) from \emptyset to λ . (Reducing the residues in (14) modulo p in the obvious way gives the formal characters of the Specht module in characteristic p .) We refer to [32] for a concise self-contained approach to the complex representation theory of S_n along the lines described here.

Now we explain how Lemma 4.12 can be used to describe some composition factors of Specht modules—this provides new non-trivial information on decomposition numbers which is difficult to obtain by other methods. The following result follows easily from Lemma 4.12.

Lemma 4.13. *Let M be an FS_n -module and set $\varepsilon = \varepsilon_i(M)$. If $[e_i^{(\varepsilon)}M : D^\mu] = m > 0$ then $\tilde{f}_i^\varepsilon D^\mu \neq 0$ and $[M : \tilde{f}_i^\varepsilon D^\mu] = m$.*

Example 4.14. Let $p = 3$. By [13, Tables], the composition factors of the Specht module $S^{(6,4,2,1)}$ are $D^{(12,1)}$, $D^{(9,4)}$, $D^{(9,2^2)}$, $D^{(7,4,2)}$, $D^{(6,5,2)}$, $D^{(6,4,3)}$, and $D^{(6,4,2,1)}$, all appearing with multiplicity 1. As $\varepsilon_1(S^{(6,4,2^2)}) = 1$ (by (14) reduced modulo 3) and $e_1 S^{(6,4,2^2)} = S^{(6,4,2,1)}$, application of Lemma 4.13 implies that the following composition factors appear in $S^{(6,4,2^2)}$ with multiplicity 1: $D^{(12,1^2)}$, $D^{(9,4,1)}$, $D^{(9,3,2)}$, $D^{(8,4,2)}$, $D^{(6^2,2)}$, $D^{(6,4^2)}$, and $D^{(6,4,2^2)}$.

Given $\underline{i} = (i_1, \dots, i_n) \in I^n$ we can gather consecutive equal terms to write it in the form

$$\underline{i} = (j_1^{m_1} \dots j_r^{m_r}) \quad (15)$$

where $j_s \neq j_{s+1}$ for all $1 \leq s < r$. For example $(2, 2, 2, 1, 1) = (2^3 1^2)$. Now, for an FS_n -module M , the tuple (15) is called *extremal* if

$$m_s = \varepsilon_{j_s}(e_{j_{s+1}}^{m_{s+1}} \dots e_{j_r}^{m_r} M)$$

for all $s = r, r-1, \dots, 1$. Informally speaking this means that among all the n -tuples \underline{i} such that $M[\underline{i}] \neq 0$ we first choose those with the longest

j_r -string in the end, then among these we choose the ones with the longest j_{r-1} -string preceding the j_r -string in the end, etc. By definition $M[\underline{i}] \neq 0$ if \underline{i} is extremal for M .

Example 4.15. The formal character of the Specht module $S^{(5,2)}$ in characteristic 3 is

$$e^{(0210201)} + 2e^{(0120201)} + 2e^{(0212021)} + 4e^{(012^20^21)} \\ + e^{(0212010)} + 2e^{(012^2010)} + e^{(0120210)} + e^{(0120120)}.$$

The extremal tuples are (012^20^21) , (012^2010) , (0120210) , and (0120120) .

Our main result about extremal tuples is

Theorem 4.16. *Let $\underline{i} = (i_1, \dots, i_n) = (j_1^{m_1} \dots j_r^{m_r})$ be an extremal tuple for an irreducible FS_n -module D^λ . Then $D^\lambda = \tilde{f}_{i_n} \dots \tilde{f}_{i_1} D^\emptyset$, and $\dim D^\lambda[\underline{i}] = m_1! \dots m_r!$. In particular, the tuple \underline{i} is not extremal for any irreducible $D^\mu \not\cong D^\lambda$.*

Proof. We apply induction on r . If $r = 1$, then by considering possible n -tuples appearing in the Specht module S^λ , of which D^λ is a quotient, we conclude that $n = 1$ and $D = D^{(1)}$. So for $r = 1$ the result is obvious. Let $r > 1$. By definition of an extremal tuple, $m_r = \varepsilon_{j_r}(D^\lambda)$. So, in view of Lemmas 4.6 and 4.12, we have

$$e_{j_r}^{m_r} D^\lambda = m_r! \tilde{e}_{j_r}^{m_r} D^\lambda.$$

Moreover, $(j_1^{m_1} \dots j_{r-1}^{m_{r-1}})$ is clearly an extremal tuple for the irreducible module $\tilde{e}_{j_r}^{m_r} D^\lambda$. So the inductive step follows. \square

Corollary 4.17. *If M is an FS_n -module and $\underline{i} = (i_1, \dots, i_n) = (j_1^{m_1} \dots j_r^{m_r})$ is an extremal tuple for M then the multiplicity of $D^\lambda := \tilde{f}_{i_n} \dots \tilde{f}_{i_1} D^\emptyset$ as a composition factor of M is $\dim M[\underline{i}] / (m_1! \dots m_r!)$.*

We note that for any tuple \underline{i} represented in the form (15) and any FS_n -module M we have that $\dim M[\underline{i}]$ is divisible by $m_1! \dots m_r!$. This follows from the properties of the principal series modules ('Kato modules') for degenerate affine Hecke algebras, see [11] for more details.

Example 4.18. In view of Corollary 4.17 extremal tuple (012^20^21) in Example 4.15 yields the composition factor $D^{(5,2)}$ of $S^{(5,2)}$, while the extremal tuple (0120120) yields the composition factor $D^{(7)}$. It turns out that these are exactly the composition factors of $S^{(5,2)}$, see e.g. [13, Tables].

For more non-trivial examples let us consider a couple of Specht modules for $n = 11$ in characteristic 3. For $S^{(6,3,1^2)}$, Corollary 4.17 yields composition factors $D^{(6,3,1^2)}$, $D^{(7,3,1)}$, and $D^{(8,2,1)}$ but 'misses' $D^{(11)}$, and for $S^{(4,3,2^2)}$ we get hold of $D^{(4,3,2^2)}$, $D^{(5,3,2,1)}$, $D^{(8,2,1)}$, and $D^{(8,3)}$, but 'miss' $2D^{(11)}$ and $D^{(5,4,1^2)}$, cf. [13, Tables].

We record here one other useful general fact about formal characters which follows from the Serre relations satisfied by the operators e_i :

Lemma 4.19. *Let M be an FS_n -module. Assume $i, j, i_1, \dots, i_{n-2} \in I$ and $i \neq j$.*

(i) *Assume that $|i - j| > 1$. Then for any $1 \leq r \leq n - 2$ we have*

$$\begin{aligned} & \dim M[(i_1, \dots, i_r, i, j, i_{r+1}, \dots, i_{n-2})] \\ &= \dim M[(i_1, \dots, i_r, j, i, i_{r+1}, \dots, i_{n-2})]. \end{aligned}$$

(ii) *Assume that $|i - j| = 1$ and $p > 2$. Then for any $1 \leq r \leq n - 3$ we have*

$$\begin{aligned} & 2 \dim M[(i_1, \dots, i_r, i, j, i, i_{r+1}, \dots, i_{n-3})] \\ &= \dim M[(i_1, \dots, i_r, i, i, j, i_{r+1}, \dots, i_{n-3})] \\ & \quad + \dim M[(i_1, \dots, i_r, j, i, i, i_{r+1}, \dots, i_{n-3})]. \end{aligned}$$

(iii) *Assume that $|i - j| = 1$ and $p = 2$. Then for any $1 \leq r \leq n - 4$ we have*

$$\begin{aligned} & \dim M[(i_1, \dots, i_r, i, i, i, j, i_{r+1}, \dots, i_{n-4})] \\ & \quad + 3 \dim M[(i_1, \dots, i_r, i, j, i, i, i_{r+1}, \dots, i_{n-4})] \\ &= \dim M[(i_1, \dots, i_r, j, i, i, i, i_{r+1}, \dots, i_{n-4})] \\ & \quad + 3 \dim M[(i_1, \dots, i_r, i, i, j, i, i_{r+1}, \dots, i_{n-4})]. \end{aligned}$$

4.7. Blocks. Finally we discuss some properties of blocks, assuming now that $p \neq 0$. In view of Theorem 4.7(ii), the blocks of the FS_n for all n are in 1–1 correspondence with the non-zero weight spaces of the basic module $V(\Lambda_0)$ of $\mathfrak{g} = A_{p-1}^{(1)}$. So let us begin by describing these following [16, ch.12].

Let $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ denote the weight lattice associated to \mathfrak{g} . Let α_i ($i \in I$) be the simple roots of \mathfrak{g} , defined from

$$\alpha_0 = 2\Lambda_0 - \Lambda_1 - \Lambda_{p-1} + \delta, \quad \alpha_i = 2\Lambda_i - \Lambda_{i+1} - \Lambda_{i-1} \quad (i \neq 0). \quad (16)$$

There is a positive definite symmetric bilinear form $(\cdot | \cdot)$ on $\mathbb{R} \otimes_{\mathbb{Z}} P$ with respect to which $\alpha_0, \dots, \alpha_{p-1}, \Lambda_0$ and $\Lambda_0, \dots, \Lambda_{p-1}, \delta$ form a pair of dual bases. Let W denote the Weyl group of \mathfrak{g} , the subgroup of $GL(\mathbb{R} \otimes_{\mathbb{Z}} P)$ generated by s_i ($i \in I$), where s_i is the reflection in the hyperplane orthogonal to α_i . Then, by [16, (12.6.1)], the weight spaces of $V(\Lambda_0)$ are the weights

$$\{w\Lambda_0 - d\delta \mid w \in W, d \in \mathbb{Z}_{\geq 0}\}.$$

For a weight of the form $w\Lambda_0 - d\delta$, we refer to $w\Lambda_0$ as the corresponding *maximal weight*, and d as the corresponding *depth*.

There is a more combinatorial way of thinking of the weights. Following [24, I.1, ex.8] and [14, §2.7], to a p -regular partition λ one associates the corresponding p -core $\tilde{\lambda}$ and p -weight d : $\tilde{\lambda}$ is the partition obtained from λ by successively removing as many hooks of length p from the rim of λ as possible, in such a way that at each step the diagram of a partition remains. The number of p -hooks removed is the p -weight d of λ . The p -cores are in 1–1 correspondence with the maximal weights, i.e. the weights belonging to the W -orbit $W\Lambda_0$, and the p -weight corresponds to the notion of depth

introduced in the previous paragraph, see [21, §5.3] and [22, §2] for the details.

Now Theorem 4.7(ii) gives yet another proof of the *Nakayama conjecture*: the FS_n -modules D^λ and D^μ belong to the same block if and only if λ and μ have the same p -core. We will also talk about the p -weight of a block B , namely, the p -weight of any λ such that D^λ belongs to B .

The Weyl group W acts on the \mathfrak{g} -module $R_{\mathbb{C}}$ from §4.3, the generator s_i ($i \in I$) of W acting by the familiar formula

$$s_i = \exp(-e_i) \exp(f_i) \exp(-e_i).$$

The resulting action preserves the Shapovalov form, and leaves the lattices R and R^* invariant. Moreover, W permutes the weight spaces of $R_{\mathbb{C}}$ in the same way as its defining action on the weight lattice P . Since W leaves δ invariant, it follows that the action is transitive on all weight spaces of the same depth. So using Theorem 4.7(iii) we see:

Theorem 4.20. *Let B and B' be blocks of symmetric groups with the same p -weight. Then, B and B' are isometric, in the sense that there is an isomorphism between their Grothendieck groups that is an isometry with respect to the Cartan form.*

The existence of such isometries was first noticed by Enguehard [8]. Implicit in Enguehard's paper is the following conjecture, made formally by Rickard: *blocks B and B' of symmetric groups with the same p -weight should be derived equivalent.* This has been proved by Rickard for blocks of p -weight ≤ 5 . Moreover, it is now known by work of Marcus [25] and Chuang-Kessar [6] that the famous Abelian Defect Group Conjecture of Broué for symmetric groups follows from the Rickard's conjecture above.

There is one situation that is particularly straightforward, when there is actually a *Morita* equivalence between blocks of the same p -weight. This is a theorem of Scopes [34], though we are stating the result in a more Lie theoretic way following [22, §8]:

Theorem 4.21. *Let $\Lambda, \Lambda + \alpha_i, \dots, \Lambda + r\alpha_i$ be an α_i -string of weights of $V(\Lambda_0)$ (so $\Lambda - \alpha_i$ and $\Lambda + (r+1)\alpha_i$ are not weights of $V(\Lambda_0)$). Then the functors $f_i^{(r)}$ and $e_i^{(r)}$ define mutually inverse Morita equivalences between the blocks parametrized by Λ and by $\Lambda + r\alpha_i$.*

Proof. Since $e_i^{(r)}$ and $f_i^{(r)}$ are both left and right adjoint to one another, it suffices to check that $e_i^{(r)}$ and $f_i^{(r)}$ induce mutually inverse bijections between the isomorphism classes of irreducible modules belonging to the respective blocks. This follows by Lemma 4.12. \square

Let us end the discussion with one new result here: we can in fact explicitly compute the determinant of the Cartan matrix of a block. The details of the proof will appear in [5]. Note in view of Theorem 4.20, the determinant of the Cartan matrix only depends on the p -weight of the block. Moreover, by Theorem 4.7(iii), we can work instead in terms of the Shapovalov form

on $V(\Lambda_0)$. Using the explicit construction of the latter module over \mathbb{Z} given in [7], we show:

Theorem 4.22. *Let B be a block of p -weight d of FS_n . Then the determinant of the Cartan matrix of B is p^N where*

$$N = \sum_{\lambda=(1^{r_1}2^{r_2}\dots)\vdash d} \frac{r_1 + r_2 + \dots}{p-1} \binom{p-2+r_1}{r_1} \binom{p-2+r_2}{r_2} \dots$$

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