1. Review of some basic theory

In this chapter, I want to quickly run through some basic theorems which you probably remember either from 600 algebra or from Frank's course. A good reference for a slightly more detailed review if you want it is chapter one of Benson's book "Representations and cohomology I".

(1) *The Jordan-Hölder theorem.* A composition series for an R-module M is a (finite) series of submodules

$$0 = M_0 < M_1 < \cdots < M_n = M$$

such that each M_i/M_{i-1} is *irreducible* (or *simple*). The Jordan-Holder theorem says that any two (finite) series of submodules of an arbitrary module M can always be refined to series of equal length such that the factors in one series are isomorphic to the factors in the other series, possibly up to a permutation.

(2) *Chain conditions.* A module M satisfies DCC if every descending chain of submodules eventually stops, and ACC if every ascending chain eventually stops. By the Jordan-Holder theorem, M has a composition series if and only if it satisfies both ACC and DCC, in which case any series of submodules of M can be refined to a composition series. In that case, for a simple module L, the *composition multiplicity* $[M : L]$ counting the number of factors of a composition series of M that are isomorphic to L is a well-defined invariant of M.

(3) *Noetherian rings.* A ring R is called *Noetherian* if the *regular module* ${}_RR$ satisfies ACC on submodules (a.k.a. left ideals). All the rings we'll meet in this course will be Noetherian. If R is Noetherian, every finitely generated R-module satisfies ACC. Moreover, every submodule of a finitely generated R-module is also finitely generated.

(4) *Completely reducible modules.* The *socle* of an R-module M is the sum of all the irreducible submodules of M, written $\operatorname{soc} M$. A module M is called *completely reducible* (or *semisimple*) if $M = \operatorname{soc} M$, which by Zorn's lemma is equivalent to the statement that every submodule of M has a complement, or to the statement that M is a direct sum of irreducible modules. By its definition, the socle $\operatorname{soc} M$ is the largest completely reducible submodule of M. The *radical* of M is the intersection of all the maximal submodules of M, written $\operatorname{rad} M$. If M satisfies DCC, then M is semisimple if and only if $\operatorname{rad} M = 0$. So assuming that M satisfies DCC, $M/\operatorname{rad} M$ is the largest completely reducible quotient of M.

(5) *The Jacobson radical.* The Jacobson radical $J(R)$ of R is the intersection of the maximal left ideals of R, or equivalently, the intersection of the annihilators of all the simple R-modules. The latter description makes it clear that $J(R)$ is a two-sided ideal of R. Note

tautologically that $J(R) = \mathrm{rad}({}_RR)$. We have *Nakayama's lemma*: If M is a finitely generated R-module and $J(R)M = M$ then $M = 0$.

(6) *Artinian rings.* A ring is called *Artinian* if R satisfies DCC on left ideals. In that case, (i) every finitely generated R-module has DCC on submodules; (ii) by Nakayama's lemma, $\mathrm{rad}(M) = J(R)M$ for any finitely generated M; (iii) $J(R)$ is nilpotent, i.e. $J(R)^n = 0$ for some n. Now suppose that M is a finitely generated R-module, and let $M_i = J(R)^iM$. Then, $\mathrm{rad}(M_i/M_{i+1}) = J(R)(M_i/M_{i+1}) = 0$, so M_i/M_{i+1} is completely reducible. Since M has DCC, so does M_i/M_{i+1}, hence it is a finite direct sum of irreducible modules, hence it has a composition series. Since $J(R)$ is nilpotent, $M_n = 0$ for some n, hence M itself has a composition series. This shows: if R is Artinian, then every finitely generated R-module has a composition series. Note applying this to the regular module, you get that Artinian rings are Noetherian.

(7) *Schur's lemma.* Schur's lemma says that if $M \ncong N$ are irreducible R-modules, then $\mathrm{Hom}_R(M, N) = 0$, while $\mathrm{End}_R(M, M)$ is a division ring. Thus, if M is a finite direct sum of irreducible R-modules, say $M = M_1^{\oplus n_1} \oplus \cdots \oplus M_r^{\oplus n_r}$ with $M_i \ncong M_j$, you get that $\mathrm{End}_R(M) \cong M_{n_1}(\mathrm{End}_R(M_1)) \oplus \cdots \oplus M_{n_r}(\mathrm{End}_R(M_r))$, a direct sum of matrix algebras over division rings. There is a stronger form of Schur's lemma too when R is an algebra over an algebraically closed field k and M is a finite dimensional irreducible R-module: in that case, $\mathrm{End}_R(M) = k$. Proof: take $f \in \mathrm{End}_R(M)$. Let $\lambda \in k$ be an eigenvalue. Then, $\ker(f - \lambda)$ is a non-zero R-submodule of M, so its all of M as M is irreducible. Hence, $f = \lambda$ is a scalar and $\mathrm{End}_R(M) = k$.

(8) *Wedderburn's theorem.* A ring R is called *semisimple* if $J(R) = 0$ (so $R/J(R)$ is *always* a semisimple ring!). Assuming that R is Artinian, R is semisimple if and only if every R-module is completely reducible, which is if and only if the regular module is completely reducible. So we can decompose ${}_RR = M_1^{\oplus n_1} \oplus \cdots \oplus M_r^{\oplus n_r}$ for irreducible modules $M_i \ncong M_j$, and get by Schur's lemma that

$$R \cong \mathrm{End}_R({}_RR)^{\mathrm{op}} \cong M_{n_1}(\mathrm{End}_R(M_1)) \oplus \cdots \oplus M_{n_r}(\mathrm{End}_R(M_r)),$$

a direct sum of matrix algebras over division rings. Moreover, R has exactly r different irreducible modules up to isomorphism, of dimensions $n_1, \dots, n_r$, namely the modules of column vectors over each of the matrix algebras. Hence, the numbers r, $n_1, \dots, n_r$ and the division rings $\mathrm{End}_R(M_i)$ are uniquely determined by R. Conversely, any finite direct sum of matrix algebras over division rings is a semisimple Artinian ring.

(9) *Fitting's lemma.* Suppose the R-module M has a composition series (i.e. both ACC and DCC). Let $f \in \mathrm{End}_R(M)$. Fitting's lemma says that for sufficiently large n, $M = \mathrm{im}(f^n) \oplus \ker(f^n)$. Let me

explain the main application of Fitting's lemma. An R-module is called *indecomposable* if it cannot be written as a direct sum of two non-zero submodules. A *local ring* is a ring R with a unique maximal left ideal (which must therefore be equal to its Jacobson radical since that is the intersection of all the maximal left ideals). Now I claim that *if M is an indecomposable module having a composition series, then* $\text{End}_R(M)$ *is a local ring.* Let I be a maximal left ideal of $E = \text{End}_R(M)$. Pick $a \notin I$. We need to show that $Ea = E$. Well, $E = Ea + I$, so we can write $1_E = \lambda a + \mu$ with $\lambda \in E, \mu \in I$. By Fitting's lemma $M = \text{im}(\mu^n) \oplus \ker(\mu^n)$ for some n. But M is indecomposable, so either μ^n is onto or $\mu^n = 0$. The former cannot occur since μ is not a unit, so it is not an automorphism of M. Hence, $\mu^n = 0$. Now $(1 + \mu + \cdots + \mu^{n-1})\lambda a = (1 + \cdots + \mu^{n-1})(1 - \mu) = 1 - \mu^n = 1$, so $Ea = E$. Using this you can now easily show that if R is an Artinian ring, then a finitely generated R-module M is indecomposable if and only if $\text{End}_R(M)$ is a local ring.

(10) *The Krull-Schmidt theorem.* Suppose R is Artinian. Then, every finitely generated R-module decomposes uniquely up to isomorphism as a direct sum of indecomposable modules. You can show more generally that if R is any old ring and M is an R-module that is a direct sum of finitely many indecomposables M_i such that each $\text{End}_R(M_i)$ is a local ring, then all other decompositions of M as a direct sum of indecomposables are isomorphic to the given one.

(11) *Projective modules.* Remember that a module P is called *projective* if every map from P to a quotient of a module M lifts to a map from P to M itself. This is equivalent to the statement that every map $M \twoheadrightarrow P$ splits, and that P is a summand of a free module. A *projective generator* means a finitely generated projective R-module P such that every R-module is a quotient of a direct sum of copies of P.

(12) *The Morita theorem.* Let R and S be rings. The following are equivalent: (i) R-Mod and S-Mod are equivalent categories; (ii) R-mod and S-mod are equivalent categories; (iii) there exist an R, S-bimodule ${}_RM_S$ and an S, R-bimodule ${}_SN_R$ such that $M \otimes_S N \cong R$ as an R, R-bimodule and $N \otimes_R M \cong S$ as an S, S-bimodule; (iv) there is a projective generator P for R such that $S \cong \text{End}_R(P)^{\text{op}}$. In that case, R and S are said to be *Morita equivalent*.

Exercise 2. (i) Prove that $\text{rad}(M \oplus N) = \text{rad}\, M \oplus \text{rad}\, N$.

(ii) Prove directly that the algebra $M_n(D)$ of $n \times n$ matrices over a division ring is a simple ring, i.e. it has no non-trivial two-sided ideals.

(iii) Deduce from (i) and (ii) that a finite direct sum of matrix algebras over division rings is a semisimple ring.

Exercise 3. Let M be an R-module, and let X, Y be submodules such that M/X is semisimple and M/Y is irreducible. Prove that $M/(X \cap Y)$ is semisimple. Hence prove the statement made in (4) above that if M satisfies DCC, then M is semisimple if and only if $\operatorname{rad} M = 0$.

Let me now give a few more basic examples.

Example 1.1. *Division rings.* Let D be a division ring (e.g. a field!). Since D is a simple D-module, it is a semisimple Artinian ring with just one irreducible module, namely, D itself. So every D-module is isomorphic to a direct sum of copies of D (e.g. every vector space has a basis!). Let $P = D^{\oplus n}$, a projective generator. Then, $\operatorname{End}_D(P)^{\mathrm{op}} \cong M_n(D)$ is Morita equivalent to D. The equivalence of categories between D and $M_n(D)$ is given explicitly in one direction by tensoring over D with the $M_n(D), D$-bimodule of column vectors, and in the other direction by tensoring over $M_n(D)$ with the $D, M_n(D)$-bimodule of row vectors. Thus every $M_n(D)$-module is isomorphic to a direct sum of copies of the n-dimensional module of column vectors. I stress this example because by Wedderburn's theorem, all simple Artinian rings are isomorphic to $M_n(D)$ for some division ring D.

Example 1.2. *Symmetric algebras.* I want to discuss the case of a symmetric algebra over a finite dimensional vector space. It is customary to work with the dual space... So, let $V \neq 0$ be a finite dimensional vector space over an algebraically closed field k (to make life easy). Let $x_1, \dots, x_n$ be a basis for V^*. Then, the symmetric algebra $S(V^*)$ can be identified with the polynomial ring $k[x_1, \dots, x_n]$, and we can think of its elements as functions on V. Hilbert's basis theorem says that $S(V^*)$ is Noetherian. It is *not* Artinian!!! The Nullstellensatz shows that the irreducible $S(V^*)$-modules are in 1–1 correspondence with the points in the vector space V, $v \in V$ corresponding to the one dimensional irreducible module k_v on which $f \in S(V^*)$ acts as multiplication by the scalar $f(v)$. The annihilator of the module k_v is the maximal ideal I_v of $S(V^*)$ consisting of all functions that are zero on the point v. The Jacobson radical $J(S(V))$ is the intersection of the annihilators of all the points $v \in V$, hence it is the set of all functions that are zero on all of V. By the Nullstellensatz again, that is zero. Hence, $J(S(V^*)) = 0$.

Example 1.3. *Polynomials in one variable.* You should also recall the special case that $A = k[x]$ is a polynomial ring in one variable over an algebraically closed field. In that case, A is a PID so we have an especially good theory for finitely generated modules. Indeed, if M is a finitely generated A-module, then it splits uniquely as a direct sum of a torsion part and a free part. So suppose that M is torsion. Then it is a finite dimensional vector space and the A-module structure is completely determined by the endomorphism defined by the action of x. Now you can put this endomorphism into Jordan normal form, and deduce that the indecomposable summands of M are precisely the Jordan blocks. Thus you get a complete classification of the finitely generated indecomposable modules: either $k[x]$ itself, or the $n \times n$ Jordan block $J_n(\lambda)$ of eigenvalue $\lambda \in k$. The irreducible modules are the $J_1(\lambda)$'s. Note the module $J_n(\lambda)$ is a *uniserial module*, meaning it has a unique composition series, and all the composition factors are isomorphic to $J_1(\lambda)$.

Example 1.4. *Exterior algebras.* So much for commutative algebra. What about skew-commutative algebra? Let V be a finite dimensional vector space of dimension n. Consider the exterior algebra $A = \bigwedge V = \bigoplus_{d\geq 0} \bigwedge^d V$ of dimension 2^n. Since it is finite dimensional, it is Artinian. The set of all non-units in $\bigwedge V$ is precisely the left ideal $\bigoplus_{d>0} \bigwedge^d V$. Hence this must be the unique maximal left ideal, so is the Jacobson radical, and the quotient is the field k. Since the Jacobson radical acts as zero on any completely reducible module, the irreducible modules of A are precisely the irreducible modules of $A/J(A)$. But that is the field k. Therefore there is a unique irreducible module, namely the field k itself.

Example 1.5. *Group algebras.* Let G be a finite group. Then the group algebra kG, k a field, is the algebra equal to the vector space with basis the elements of G and with multiplication given by extending the multiplication in the group G by bilinearity. Since kG is a finite dimensional algebra, it is Artinian. You probably remember *Maschke's theorem*: the algebra kG is semisimple if and only if $\operatorname{char} k \nmid |G|$. Since it is so important, let's run through the proof.

Suppose that $\operatorname{char} k \nmid |G|$. Let M be a kG-module and let N be a submodule. Let $\pi : M \twoheadrightarrow N$ be any linear map extending the identity map on N. For $g \in G$, consider $g^{-1} \circ \pi \circ g : M \to N$. It also extends the identity map on N. Hence so does

$$\frac{1}{|G|} \sum_{g\in G} g^{-1} \circ \pi \circ g.$$

But that is now even G-equivariant. Hence its kernel is a G-stable complement to N in M. We've shown every submodule of a G-module has a complement, which means kG is a semisimple algebra. Conversely, suppose that $\operatorname{char} k = p||G|$. Let $e = \sum_{g\in G} g$. Since $ge = e = eg$ for each $g \in G$, e spans a one dimensional ideal in kG. Since $e^2 = 0$, this ideal is nilpotent, so it is contained in the Jacobson radical because in an Artinian ring, $J(R)$ is the sum of all the nilpotent ideals of R. Hence, $J(kG) \neq 0$ and kG is not semisimple.

By the way, when talking about kG-modules, people often use an alternative language and call a kG-module M instead a *representation of G*. That is because the action of G on the module M induces a group homomorphism $\rho : G \to GL(M)$ which "represents" the group as a group of invertible $\dim M \times \dim M$ matrices.

Assume from now on that k is algebraically closed of characteristic 0. By Schur's lemma (in its strong form for finite dimensional modules and an algebraically closed field) the endomorphism algebra of a simple module is just k. So by Wedderburn's theorem, $kG = M_{n_1}(k) \oplus \cdots \oplus M_{n_r}(k)$, where the number r is the number of inequivalent irreducible representations, and

$n_1, \dots, n_r$ are the dimensions of the respective simple modules. Question: what is r exactly? Well, consider the center $Z(kG)$. It is r-dimensional, since you have the scalar matrices in each $M_{n_i}(k)$. On the other hand, an easy calculation shows that if $\sum_{g \in G} c_g g \in kG$ is a central element, then the coefficients c_g must be constant on each conjugacy class of G. Hence, the dimension of the center $Z(kG)$ is equal to the number of conjugacy classes in G. Therefore: *the number of inequivalent irreducible kG-modules is equal to the number of conjugacy classes in G.* Moreover, their dimensions $n_1, \dots, n_r$ satisfy

$$|G| = n_1^2 + \cdots + n_r^2.$$

This is the starting point for character theory of finite groups, which provides many more wonderful numerical connections between the structure of the group and its representations.

Example 1.6. *Abelian and cyclic groups.* Let G be a finite abelian group and k be an algebraically closed field of characteristic 0. We've just seen that there are $|G|$ inequivalent irreducible representations, and they must all be one dimensional.

Take for instance $G = C_n$, a cyclic group. Then we can easily construct all the one dimensional irreducibles, as follows. Let $g \in G$ be a generator. Let ω be a primitive nth root of unity in k. Then the rth irreducible representation is the field k on which g acts as the scalar ω^r, for $r = 0, 1, \dots, n-1$.

There's another way to see this: the group algebra kC_n is the quotient of the polynomial algebra $k[x]$ by the ideal $(x^n - 1)$. Since $(x^n - 1) = (x-1)(x-\omega)\dots(x-\omega^{n-1})$ and these are relatively prime factors, the Chinese Remainder Theorem shows that

$$kC_n \cong k[x]/(x-1) \oplus \cdots \oplus k[x]/(x-\omega^{n-1}).$$

Hence we've decomposed the group algebra as a direct sum of 1×1 matrix algebras! This approach lets you get a glimpse of what happens when the field is not algebraically closed: it is all about how $(x^n - 1)$ can be factorized over your ground field.

Exercise 4. Classify the indecomposable modules of the group algebra kC_n of the cyclic group of order n over an algebraically closed field of characteristic p.