

TOPICS IN REPRESENTATION THEORY

Representation theory is a rather wide subject, with many different points of view. One natural place to start in a course on representation theory is with character theory of finite groups. I am *not* going to do this – though you are missing a lot! Instead I will start with some representation theory of finite dimensional algebras and quivers. Then I am going to switch to doing representation theory of Lie algebras, which has a nice character theory of its own. Hopefully I’ll be able to make some of the connections between the two subjects...

I will set exercises as we go, and collect them in every couple of weeks or so (usually when I get to 10!). This will probably happen about 4 times throughout the term. You really should aim to do the exercises as soon as possible after they have been set, since they are meant to prepare you for following what is going on in lectures. The first exercise is on this first set of notes!!!

Conventions.

- (1) All rings will be associative and unital (but usually *not* commutative). All modules will be unital modules.
- (2) Given a ring R , write $R\text{-Mod}$ for the category of all left R -modules, and $R\text{-mod}$ for the category of all *finitely generated* left R -modules.
- (3) Usually we’ll have in mind a *ground field* k and then our rings will always actually be k -algebras. I’ll often just say “let A be an algebra” meaning an algebra over the field k . For example, a *finite dimensional algebra* means a finite dimensional vector space over k together with a bilinear multiplication that makes A into a ring. Usually, k will be algebraically closed to avoid ever having to think about field extensions.
- (4) Note if A is a finite dimensional algebra, then $A\text{-mod}$ is equivalently the category of all finite dimensional left A -modules.
- (5) You should remember that $\text{Hom}_A({}_A A, {}_A M)$ is canonically isomorphic to ${}_A M$ as a left A -module, the isomorphism being given by evaluation at 1. Similarly, $\text{Hom}_A(A_A, M_A)$ is canonically isomorphic to M_A as a right A -module. Where is the module structure on the hom coming from?
- (6) I’m almost always going to consider left modules, so from now on when I say A -module, I’ll mean left A -module. Also I’m always going to write maps on the left. So $\text{End}_A(A) = \text{Hom}_A({}_A A, {}_A A)$ is canonically isomorphic to A – but this is not as an algebra because it has the opposite multiplication. We’ll have to write $A \cong \text{End}_A(A)^{\text{op}}$

to get a correct statement as an algebra – which is the penalty for writing maps on the left.

The first exercise should check you understand my conventions...

Exercise 1. Let M be an R -module and let $e \in R$ be an idempotent (i.e. $e^2 = e$). Then, eRe is a subring of R (but it is not a unital subring). Show that

$$eM \cong \text{Hom}_R(Re, M)$$

as eRe -modules and that

$$eRe \cong \text{End}_R(Re)^{\text{op}}$$

as rings.