393 Homework 4 solutions

Exercises 6.1: 0, 1,2,4, 10, 11, 14.

0 An abelian grape. (This is famous as it is the ONLY mathematical joke). Sorry.

1 (a) YES it is the group $\mathbb{Z}_{10}^{\times}$ of all units in $\mathbb{Z}_{10}$. (b) NO not closed under + (need 8). (c) NO for instance the inverse of 1/2 would be 2 which is not in the set. (d) NO the unit would have to be 1 which is rational. (e) YES this is isomorphic to the group $(\mathbb{R}, +)$ of real numbers under addition. (f) YES this is usually known as the circle group S^1. (g) YES: $e = -1$, $a^{-1} = -a - 2$ and it is associative (check). (h) NO it is not associative: $(a \circ b) \circ c = (a - b) - c = a - b - c$ while $a \circ (b \circ c) = a - (b - c) = a - b + c$. (i) YES: $e = 0$, $a^{-1} = a/(a-1)$ (which makes sense as $a \neq 1$) and it is associative (check).

2 (a) All subgroups of $\mathbb{Z}_{18}$: there is exactly one for each divisor of 18. The subgroups are $\langle 1 \rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\}$, $\langle 2 \rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}$, $\langle 3 \rangle = \{0, 3, 6, 9, 12, 15\}$, $\langle 6 \rangle = \{0, 6, 12\}$, $\langle 9 \rangle = \{0, 9\}$ and $\langle 0 \rangle = \{0\}$.

(b) All subgroups of $\mathbb{Z}_5$: only $\{0\}$ and $\mathbb{Z}_5$ itself.

(c) All subgroups of $\mathbb{Z}_5^{\times}$: this group is isomorphic to the group $(\mathbb{Z}_4, +)$ so there are three subgroups as three divisors of 4. They are $\langle 1 \rangle = \{1\}$, $\langle 2 \rangle = \{1, 2, 3, 4\} = \langle 3 \rangle$ and $\langle 4 \rangle = \{1, 4\}$.

(d) All subgroups of $\mathbb{Z}_{11}^{\times}$: this group is isomorphic to the group $(\mathbb{Z}_{10}, +)$. Since the order of $2 \in \mathbb{Z}_{11}^{\times}$ is 10, the isomorphism maps 2^n to n. We know the subgroups of $\mathbb{Z}_{10}$: $\langle 1 \rangle, \langle 2 \rangle, \langle 5 \rangle, \langle 0 \rangle$. These then give us the subgroups of $\mathbb{Z}_{11}^{\times}$: $\langle 2 \rangle, \langle 4 \rangle, \langle 10 \rangle, \langle 1 \rangle$.

4 Let $GL(2, \mathbb{Z}_2)$ be the group of invertible 2×2 matrices with entries in $\mathbb{Z}_2$. List its elements. What is the order of the group? What are its subgroups?

The elements are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

So it is a group of order 6. In fact it is isomorphic to the group of symmetries of the equilaterial triangule (but you have to work out the multiplication table in order to prove this). It has subgroups $\{e\}$, the whole thing, three subgroups of order 2 and one subgroup of order 3 – 6 subgroups in all.

10 (a) Let G be a group. Prove that $(ab)^2 = a^2b^2$ for all $a, b \in G$ if and only if G is abelian. (b) Prove that if every element (other than the identity element) of a group G has order 2 then G is abelian.

Proof. *(a) If G is abelian then $(ab)^2 = abab = aabb = a^2b^2$. Conversely, if $(ab)^2 = a^2b^2$ then $abab = aabb$ so multiplying on the left by a^{-1} and on the right by b^{-1} gives that $ab = ba$, so it is abelian.*

(b) $(ab)^2 = e$. *Hence* $abab = e$. *Multiplying on the left by* a^{-1} *and on the right by* b^{-1} *gives that* $ba = ab$. *Hence it is abelian.*

11 Prove that any group of order ≤ 4 is abelian.

This follows from what we did in class: we computed the multiplication tables of all the groups of order ≤ 4 *and they were all abelian.*

14 Show that the group of symmetries of a rectangle is the Klein 4-group.

There are just four symmetries, e *(the identity),* a *(reflection in the vertical axis),* b *(reflection in the horizontal axis), and* r *(rotation through 180 degrees). Now check that* $ab = r = ba, ar = b = ra, br = a = rb$ *and* $a^2 = b^2 = r^2 = 1$. *So it has the same multiplication table as the Klein 4 group.*