

393 HOMEWORK 2 SOLUTIONS

• Exercises 5.1: 13,15,20.

13 Suppose that $[K : F]$ is prime. Show that if $\alpha \in K$ and $\alpha \notin F$ then $K = F[\alpha]$.

We'll use the basic fact that $[K : F] = 1$ if and only if $K = F$. Consider the tower $K \supseteq F[\alpha] \supseteq F$. By the tower law, $[K : F] = p = [K : F[\alpha]][F[\alpha] : F]$. Since p is prime we must either have that $[K : F[\alpha]] = 1$ or $[F[\alpha] : F] = 1$. In the former case we get that $K = F[\alpha]$ which is what we want. In the latter case we get that $F[\alpha] = F$, which is not the case since $\alpha \notin F$.

15 (a) Prove that $[\mathbb{Q}[\sqrt[3]{2}, \sqrt[40]{2}] : \mathbb{Q}] \subseteq [\mathbb{Q}[\sqrt[120]{2}] : \mathbb{Q}]$. (b) Deduce that $\sqrt[3]{2} \notin \mathbb{Q}[\sqrt[40]{2}]$. (c) Use degree to prove that in fact $[\mathbb{Q}[\sqrt[3]{2}, \sqrt[40]{2}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt[120]{2}] : \mathbb{Q}]$.

(a) *This holds because $(\sqrt[120]{2})^{40} = \sqrt[3]{2}$ and $(\sqrt[120]{2})^3 = \sqrt[40]{2}$.*

(b) *Suppose that $\sqrt[3]{2} \in \mathbb{Q}[\sqrt[40]{2}]$. Then we have a tower*

$$\mathbb{Q} \subset \mathbb{Q}[\sqrt[3]{2}] \subset \mathbb{Q}[\sqrt[40]{2}].$$

By the tower law it follows that $[\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}]$ divides $[\mathbb{Q}[\sqrt[40]{2}] : \mathbb{Q}]$. But $3 \nmid 40$ so this is a contradiction.

(c) *In view of (a) and the tower law, it suffices to show that $[\mathbb{Q}[\sqrt[3]{2}, \sqrt[40]{2}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt[120]{2}] : \mathbb{Q}]$. The right hand side is 120. The left hand side is ≤ 120 and divisible by both 3 and 40 (using the tower law like we usually do!). Hence the left hand side is 120 too.*

20 Let W and Z be subspaces of V . (a) Prove that $W+Z$ and $W \cap Z$ are subspaces. (b) Prove that if W and Z are finite dimensional, so are $W + Z$ and $W \cap Z$. (c) Prove that $\dim(W + Z) = \dim W + \dim Z - \dim(W \cap Z)$.

(a) *You have to check they contain zero, are closed under plus and closed under scalars.*

(b) *$W \cap Z$ is a subspace of W . Subspaces of finite dimensional vector spaces are finite dimensional. Hence $W \cap Z$ is finite dimensional.*

If w_1, \dots, w_n span W and z_1, \dots, z_m span Z then

$$w_1, \dots, w_n, z_1, \dots, z_m$$

span $W + Z$. Hence $W + Z$ is finite dimensional.

(c) *Let x_1, \dots, x_a be a basis for $W \cap Z$. Extend to a basis $x_1, \dots, x_a, y_1, \dots, y_b$ for W and to a basis $x_1, \dots, x_a, z_1, \dots, z_c$ for Z . Now I claim that $x_1, \dots, x_a, y_1, \dots, y_b, z_1, \dots, z_c$ is a basis for $W + Z$.*

They obviously span. To see that they are linearly independent, suppose for a contradiction that they are dependent. Then since $x_1, \dots, x_a, y_1, \dots, y_b$ are independent already, we deduce

that there is some $0 \neq z \in \text{span}(z_1, \dots, z_c)$ that already lies in $\text{span}(x_1, \dots, x_a, y_1, \dots, y + b)$. But then $z \in Z \cap W$, so z is actually in $\text{span}(x_1, \dots, x_a)$. That shows that the vectors $x_1, \dots, x_a, z_1, \dots, z_c$ are dependent, a contradiction. This proves the claim.

Hence,

$$\dim(W + Z) = a + b + c = (a + b) + (a + c) - a = \dim W + \dim Z - \dim(W \cap Z).$$

- Exercises 5.2: 4, 6, 18.

4 I think I showed you these constructions in class. I can't type them in!

6 Again its impossible to type – ask me if you want to know how to do it. Basically to draw the in circle you have to find the point that is equidistant from each of the edges of the triangle, so you bisect the three angles in the triangle and see where they meet. To draw the out circle you have to find the point that is equidistant from each of the corners of the triangle, so you draw the perpendicular bisectors of each of the edges and see where they meet. It is good fun in either case to prove that the three lines really are concurrent (it doesn't work for example for quadrilaterals in general)!

18 I can't possibly type this one in. I should do it in class.