393 Homework 1 solutions

Exercises 5.1: 3,4,5,6,7,8,9,10,11(a)(b)(c)(d).

3 Let V be a vector space over $\mathbb{Q}$. Prove that if $v, w \in V$ are linearly independent, so are $v+w$ and $2v-w$.

Proof. *Say $a(v+w)+b(2v-w)=0$ for some $a, b \in \mathbb{Q}$. We need to show that $a=b=0$.*

Well, we have that $(a+2b)v+(a-b)w=0$. Since v and w are linearly independent that shows that $a+2b=a-b=0$. Now solving this system of linear equations for a and b gives that $a=b=0$.

4 Prove that the real numbers 1 and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$. Do the same for $1, \sqrt{3}$ and $\sqrt{5}$.

Since $\sqrt{3}$ is irrational, it is not a rational multiple of 1. Hence, $\sqrt{3} \notin span(1)$. Hence, 1 and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$.

To show $1, \sqrt{3}$ and $\sqrt{5}$ are too, it suffices to show that $\sqrt{5} \notin span(1, \sqrt{3})$. Well, suppose

$$\sqrt{5} = a + b\sqrt{3}$$

for rationals a, b. Note we cannot have $b=0$ since $\sqrt{5}$ is irrational, and we can't have $a=0$ since that would give $b=\sqrt{\frac{5}{3}}$ and $\sqrt{\frac{5}{3}}$ is also irrational. Hence, $ab \neq 0$. Then,

$$5 = a^2 + 3b^2 + 2ab\sqrt{3}.$$

Rearranging, which we can do since $ab \neq 0$, gives that

$$\sqrt{3} = \frac{5 - a^2 - 3b^2}{2ab}.$$

But that is a contradication since $\sqrt{3}$ is irrational.

5 Suppose $v_1, \dots, v_k$ are linearly independent and $v \in V$. Prove that $v_1, \dots, v_k, v$ are linearly independent if and only if $v \notin span(v_1, \dots, v_k)$.

Proof. *If $v \in span(v_1, \dots, v_k)$ then $v_1, \dots, v_k, v$ are linearly dependent, since one of them is a linear combination of the rest.*

Conversely, suppose that $v \notin span(v_1, \dots, v_k)$. To show that $v_1, \dots, v_k, v$ are linearly independent, take a linear combination

$$\sum_{i=1}^{k} \lambda_i v_i + \mu v = 0.$$

We need to show that $\lambda_1 = \dots = \lambda_k = \mu = 0$. Well, if $\mu = 0$, then since the λ_i's are linearly independent, we certainly get that $\lambda_1 = \dots = \lambda_k = 0$ already. Otherwise, if $\mu \neq 0$, we can rearrange to write

$$v = \sum_{i=1}^{k} (-\frac{\lambda_i}{\mu} v_i).$$

But that is a contradiction since v is not a linear combination of the v_i's. So this case doesn't happen.

6 Prove that $v_1, \dots, v_k$ are linearly dependent if and only if for some j, v_j is a linear combination of $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k$.

Proof. *We need to be careful with this one with what we are taking to be the definition of linear independent: remember $v_1, \dots, v_k$ are linearly independent if $\sum_{i=1}^k \lambda_i v_i = 0$ implies $\lambda_1 = \cdots = \lambda_k = 0$.*

Hence, if $v_1, \dots, v_k$ are linearly dependent, we have that

$$\sum_{i=1}^k \lambda_i v_i = 0$$

for scalars $\lambda_1, \dots, \lambda_k$ where at least one $\lambda_j \neq 0$. But then we can rearrange it as

$$v_j = -\frac{\lambda_1}{\lambda_j} v_1 - \cdots - \frac{\lambda_{j-1}}{\lambda_j} v_{j-1} - \frac{\lambda_{j+1}}{\lambda_j} v_{j+1} - \cdots - \frac{\lambda_k}{\lambda_j} v_k$$

and we have rewritten v_j as a linear combination of the rest for some j.

Conversely, if v_j is a linear combination of the rest, say

$$v_j = \lambda_1 v_1 + \cdots + \lambda_{j-1} v_{j-1} + \lambda_{j+1} v_{j+1} + \cdots + \lambda_k v_k,$$

then bringing everything over to the same side gives that

$$\lambda_1 v_1 + \cdots + \lambda_{j-1} v_{j-1} - v_j + \lambda_{j+1} v_{j+1} + \cdots + \lambda_k v_k = 0$$

But this is a linear combination equal to zero with not all the coefficients zero (because the v_j coefficient is -1). Hence they are linearly dependent.

7 Suppose $v \in span(v_1, \dots, v_k)$. Prove that $span(v_1, \dots, v_k, v) = span(v_1, \dots, v_k)$.

Proof. *Obviously, $span(v_1, \dots, v_k) \subseteq span(v_1, \dots, v_k, v)$. Conversely, take any $x \in span(v_1, \dots, v_k, v)$. We need to show that $x \in span(v_1, \dots, v_k)$ already.*

Write

$$x = \sum_{i=1}^k \lambda_i v_i + \mu v.$$

Also write

$$v = \sum_{i=1}^k \mu_i v_i.$$

Then, we get that

$$x = \sum_{i=1}^k (\lambda_i + \mu \mu_i) v_i.$$

Hence $x \in span(v_1, \dots, v_k)$ as required.

8 Prove that $v_1, \dots, v_k$ are a basis for V if and only if every vector of V can be written *uniquely* as a linear combination of $v_1, \dots, v_k$.

Proof. *Suppose $v_1, \dots, v_k$ is a basis for V. Then, any vector of V is a linear combination of them since they span. For uniqueness, suppose we have written a vector as a linear combination of them in two different ways:*

$$x = \sum_{i=1}^k \lambda_i v_i = \sum_{i=1}^k \mu_i v_i.$$

We need to show that $\lambda_i = \mu_i$ for each i. Well,

$$0 = x - x = \sum_{i=1}^k \lambda_i v_i - \sum_{i=1}^k \mu_i v_i = \sum_{i=1}^k (\lambda_i - \mu_i) v_i.$$

Since the v_i are linearly independent we get that all the coefficients $\lambda_i - \mu_i$ are zero. QED.

9 Let $v_1, \dots, v_k$ be linearly independent vectors in a finite dimensional vector space. Show that there are vectors $v_{k+1}, \dots, v_n$ such that

$$v_1, \dots, v_k, v_{k+1}, \dots, v_n$$

form a basis for V.

Proof. *If $v_1, \dots, v_k$ already span V its a basis and we're done. Else, we can find a vector $v_{k+1} \in V$ with $v_{k+1} \notin span(v_1, \dots, v_k)$. By 5, $v_1, \dots, v_k, v_{k+1}$ are linearly independent.*

If $v_1, \dots, v_{k+1}$ span V its a basis and we're done. Else we can find a vector $v_{k+2} \in V$ with $v_{k+2} \notin span(v_1, \dots, v_{k+1})$. By 5, $v_1, \dots, v_{k+2}$ are linearly independent.

Keep going.

The process stops, proving the theorem, in finitely many steps, since in a vector space of dimension n and set of $> n$ vectors is not linearly independent.

10 Let V be n dimensional over F. (a) Prove that if $v_1, \dots, v_n \in V$ are linearly independent then they span. (b) Prove that if $v_1, \dots, v_n \in V$ span then they are linearly independent.

Proof. *(a) If they do not span, we can pick $v \in V$ with $v \notin span(v_1, \dots, v_n)$. Then by 5, $v_1, \dots, v_n, v$ are again linearly independent. But in an n dimensional vector space, $n+1$ vectors can never be independent!*

(b) If they are linearly dependent, by 6 some v_j can be expressed as a linear combination of the rest. Hence by 7, $span(v_1, \dots, v_n) = span(v_1, \dots, \hat{v}_j, \dots, v_n)$. So we've got a spanning set with just $n-1$ vectors in them. But in an n dimensional vector space, $n-1$ vectors can never span!

11(a) Give a basis for $\mathbb{Q}[\sqrt{2}]$ over $\mathbb{Q}$. *Answer:* 1, $\sqrt{2}$

11(b) Give a basis for $\mathbb{Q}[i\sqrt{3}]$ over $\mathbb{Q}$. *Answer:* 1, $i\sqrt{3}$

11(c) Give a basis for $\mathbb{Q}[\sqrt{3}, i]$ over $\mathbb{Q}$. *Answer:* 1, $\sqrt{3}$, i, $i\sqrt{3}$ *(its 4 dimensional)*

11(d) Give a basis for $\mathbb{Q}[\sqrt{3}, i]$ over $\mathbb{Q}[i\sqrt{3}]$. *Answer: Notice* $\mathbb{Q}[\sqrt{3}, i] = \mathbb{Q}[i\sqrt{3}][i]$. *Hence a basis is $1, i$ and it is 2 dimensional.*