Elementary Abstract Algebra II Practise Midterm

Answer questions 1 through 4. Each of these questions is worth TWELVE points. IF you finish that you might want to attempt the bonus question 5 which is worth TWO extra points.
Show all your work and try to be as CLEAR as possible in explaining proofs. That way, I can give some credit even for wrong answers. If you're not immediately sure how to do a problem, move on to the next one then come back to it at the end!
(THE REAL MIDTERM WILL BE ON EXACTLY THE SAME TOPICS, BASICALLY EVERYTHING WE DID IN CLASS FROM CHAPTER 9 OF THE TEXT BOOK, THOUGH I TRIED TO MAKE THE PROBLEMS EASIER THAN THE ONES IN THIS PRACTISE.)

1. (a) What is a Principal Ideal Domain?

(b) Prove that if R is a Euclidean domain (with degree function δ) then R is a principal ideal domain.

2. (a) For each of the following rings, write down the definition of a degree function δ making the given ring into a Euclidean domain:
(i) $\mathbb{Z}$.

(ii) $\mathbb{C}[x]$.

(iii) $\mathbb{Z}[i]$.

(iv) $\mathbb{Q}$.

(b) If R is a Euclidean domain with degree function δ, prove that

$$R^{\times} = \{x \in R \mid \delta(x) = \delta(1)\}.$$

(c) Using (a)(iii) and (b), find all the units in the ring $\mathbb{Z}[i]$ of Gaussian integers.

3. Let R be an integral domain.
(a) Define an *irreducible element* $p \in R$.

(b) Suppose R has the property that for every pair of non-zero elements $a, b \in R$, the greatest common divisor of a and b exists and can be expressed as an R-linear combination of a and b. Let $p \in R$ be an irreducible element and $a, b \in R$ be non-zero elements. Prove that if $p|ab$ then either $p|a$ or $p|b$.

(c) Suppose R is a Unique Factorization Domain. Let $p \in R$ be an irreducible element and $a, b \in R$ be non-zero elements. Prove that if $p|ab$ then either $p|a$ or $p|b$. (*Warning*: there exist UFDs that do NOT have the property formulated in part (b)!)

4. Consider the ring $\mathbb{Z}[\sqrt{10}]$.
(a) By considering the multiplicative function

$$\delta : \mathbb{Z}[\sqrt{10}] \to \mathbb{Z},\ \ x + y\sqrt{10} \mapsto x^2 - 10y^2,$$

prove that $2, 3$ and $\sqrt{10} \pm 2$ are all irreducible elements. (Did you remember to show they not units?)

(b) By considering the factorizations $6 = 2.3 = (\sqrt{10} - 2)(\sqrt{10} + 2)$ explain why $\mathbb{Z}[\sqrt{10}]$ is *not* a Unique Factorization Domain.

!!IN THE REAL MIDTERM THERE WILL BE AN EXTRA CREDIT PROBLEM HERE!!

5. Let R be a principal ideal domain and $x \in R$ be a non-zero element. Prove that x is irreducible if and only if $R/\langle x \rangle$ is a field.