Elementary Abstract Algebra II Practise Midterm

Answer questions 1 through 4. Each of these questions is worth TWELVE points. IF you finish that you might want to attempt the bonus question 5 which is worth TWO extra points.
Show all your work and try to be as CLEAR as possible in explaining proofs. That way, I can give some credit even for wrong answers. If you're not immediately sure how to do a problem, move on to the next one then come back to it at the end!
(THE REAL MIDTERM WILL BE ON EXACTLY THE SAME TOPICS, BASICALLY EVERYTHING FROM CHAPTER 5 OF THE TEXT BOOK, THOUGH I TRIED TO MAKE THE PROBLEMS EASIER THAN THE ONES IN THIS PRACTISE. Of course you are also meant to remember everything from last term! There will be nothing on ED's or PID's.)

1. (a) Let $\sim$ be a relation on a set $S$. Write down the three axioms that make $\sim$ into an *equivalence relation.*

Reflexive : $a \sim a$
Symmetric : $a \sim b \Rightarrow b \sim a$
Transitive : $a \sim b,\ b \sim c \Rightarrow a \sim c$

(b) For each of the following, determine if the given relation is an equivalence relation, explaining your answer.
(i) For $a, b \in \mathbb{R}$ define $a \sim b$ if $a \leq b$.

NOPE eg $1 \leq 2$ but $2 \not\leq 1$ (fails symmetric)

(ii) For $a, b \in \mathbb{R}$ define $a \sim b$ if $a - b \in \mathbb{Q}$.

YUP
$a - a = 0$ ✓ Reflexive
If $a - b \in \mathbb{Q}$ then $b - a \in \mathbb{Q}$ ✓ Symmetric
If $a - b \in \mathbb{Q}$ and $b - c \in \mathbb{Q}$ then $a - c = (a-b) + (b-c) \in \mathbb{Q}$ ✓ Transitive

(iii) For $a, b \in \mathbb{R}$ define $a \sim b$ if $|a - b| \leq 1$.

NOPE eg $0 \sim 1$, $1 \sim 2$ but $0 \not\sim 2$ (fails transitive)

2. Let $\phi : \mathbb{Q}[x] \to \mathbb{R}$ be the homomorphism with $\phi(f(x)) = f(\sqrt[3]{5})$.
(a) Carefully prove that $\ker \phi = \langle x^3 - 5 \rangle$.

As $\mathbb{Q}[x]$ is a PID, $\ker \varphi = \langle f(x) \rangle$ some monic $f(x)$
As $\varphi(x^3-5) = 0$, $x^3 - 5 \in \langle f(x) \rangle$ so $f(x) \mid x^3 - 5$
As $x^3 - 5$ is irreducible by Eisenstein, must actually have that $f(x) = x^3 - 5$

(b) Define $\mathbb{Q}(\sqrt[3]{5})$ to be the following subset of $\mathbb{R}$:

$$\{a + b\sqrt[3]{5} + c\sqrt[3]{25} \mid a, b, c \in \mathbb{Q}\}.$$

Prove that this subset is actually a *subring* of $\mathbb{R}$. Start by listing the properties you need to check!

Closed under $+, -, \cdot$, contains 1 ← obvious
(boring)

$(a + b\sqrt[3]{5} + c\sqrt[3]{25})(x + y\sqrt[3]{5} + z\sqrt[3]{25}) = (ax + 5bz + 5cy) + (ay + bx + 5cz)\sqrt[3]{5} + (az + by + cx)\sqrt[3]{25}$ ✓

(c) Use the 1st isomorphism theorem to prove that $\mathbb{Q}[x]/\langle x^3 - 5 \rangle \cong \mathbb{Q}(\sqrt[3]{5})$.

$\ker \varphi = \langle x^3 - 5 \rangle$ $\quad \operatorname{Im} \varphi = \{ f(\sqrt[3]{5}) \mid f(x) \in \mathbb{Q}[x] \}$
$= \{ a + b\sqrt[3]{5} + c\sqrt[3]{25} \mid a, b, c \in \mathbb{Q} \}$ as $(\sqrt[3]{5})^3 \in \mathbb{Q}$
$= \mathbb{Q}(\sqrt[3]{5})$

$\therefore \mathbb{Q}[x]/\langle x^3 - 5 \rangle \cong \mathbb{Q}(\sqrt[3]{5})$ by 1st $\cong$ theorem

(d) Now prove that $\mathbb{Q}(\sqrt[3]{5})$ is a *field*.

LHS is a field as $x^3 - 5$ is irreducible (see (a))

$\therefore$ RHS is a field

3. Let $f : R \to R$ be an automorphism and $I$ be an ideal of $R$. Let $J = f(I) = \{f(a) \mid a \in R\}$.
(a) Prove that $J$ is an ideal of $R$. Start by listing the properties you need to check!

Closed under $+$, Extra-closed under $\cdot$, contains $0$ ⟵ $f(0)=0$ ✓

$a,b \in I$, $r \in R$: $f(a)+f(b) = f(\underbrace{a+b}_{\in I})$ ✓ $\qquad f(a)r = f(\underbrace{a f^{-1}(r)}_{\in I})$ ✓

(b) Prove that the function $\phi : R/I \to R/J, a+I \mapsto f(a)+J$ is a well-defined function.

$a+I = b+I \iff a-b \in I \iff f(a-b) \in J$
$\iff f(a)+J = f(b)+J \iff \varphi(a+I) = \varphi(b+I)$

$(\Rightarrow)$ is well-defined!

(c) Now prove that $R/I \cong R/J$.

$(\Leftarrow)$ in (b) is 1-1.

onto: take any $b+J \in R/J$. It's $\varphi(f^{-1}(b)+I)$. So onto ✓

homomorphism: additive, $1 \mapsto 1$ : boring
multiplicative: $\varphi((a+I)(b+I)) = \varphi(ab+I) = f(ab)+J$ ✓
$\varphi(a+I)\varphi(b+I) = (f(a)+J)(f(b)+J) = f(a)f(b)+J$

(d) Deduce that $\mathbb{Z}[i]/\langle x+iy\rangle \cong \mathbb{Z}[i]/\langle x-iy\rangle$ for any $x, y \in \mathbb{Z}$.

Apply (c) to $f : \mathbb{Z}[i] \to \mathbb{Z}[i]$, $x+iy \mapsto x-iy$ (complex conjugation)

4. True or False? Explain!

(a) $\mathbb{Q}(\sqrt{2})/\langle 1+\sqrt{2}\rangle \cong \mathbb{Q}$.

$\mathbb{Q}(\sqrt{2})$ is a field. In fields there are only two ideals, zero and everything.

So $\langle 1+\sqrt{2}\rangle$ = everything.

So its $\mathbb{Q}(\sqrt{2})/\mathbb{Q}(\sqrt{2})$ = the zero ring $\neq \mathbb{Q}$ (F)

(b) Every ideal in the ring $\mathbb{C}[x, y]$ is principal.

(F) eg $\langle x,y\rangle$ = all $\mathbb{C}[x,y]$-linear combinations of $x$ and $y$ is not a principal ideal

(if it was, say $\langle f(x,y)\rangle$, then $f(x,y)\mid x$ and $f(x,y)\mid y$ only such $f$ is a unit, so $\langle f(x,y)\rangle = \mathbb{C}[x,y] \neq \langle x,y\rangle$ #)

(c) If $S$ is a subring of a ring $R$ and $I$ is an ideal of $R$, then $I \cap S$ is an ideal of $S$.

(T) Take $s \in I \cap S$, $y \in S$. Need to show $sy \in I \cap S$.

It's in $S$ as $S$ is closed under $\cdot$.

It's in $I$ as $I$ is extra closed under $\cdot$.

$\therefore sy \in I \cap S$ ✓

(d) If $R$ and $S$ are integral domains then so is $R \times S$.

(F) eg $\mathbb{Z} \times \mathbb{Z}$ $(1,0)\cdot(0,1) = (0,0)$

$\therefore$ Not an integral domain.

(e) The field of fractions of $\mathbb{Z}[i]$ is isomorphic to $\mathbb{Q}(i)$.

(T) $\mathbb{Z}[i]$ is fractions $\frac{a+ib}{c+id}$ $a,b,c,d \in \mathbb{Z}$

$= \frac{(a+ib)(c-id)}{c^2+d^2} = \underbrace{\frac{ac+bd}{c^2+d^2}}_{\in \mathbb{Q}} + i\underbrace{\frac{ac-bd}{c^2+d^2}}_{\in \mathbb{Q}}$

$\mathbb{Q}(i)$ is $\frac{a}{b} + \frac{c}{d}i = \frac{ad+bci}{bd}$ $\begin{matrix}\in \mathbb{Z}[i] \\ \in \mathbb{Z}[i]\end{matrix}$

This lets you write down maps

$\mathbb{Q}(i) \leftrightarrows$ f.o.f. $(\mathbb{Z}[i])$

... isomorphisms

!!IN THE REAL MIDTERM THERE WILL BE AN EXTRA CREDIT PROBLEM HERE!!

5. Let $\sim$ be the equivalence relation on $\mathbb{R}$ defined by $a \sim b$ if $a - b \in \mathbb{Q}$. Define operations $+$ and $\cdot$ on the set $\mathbb{R}/\sim$ of $\sim$-equivalence classes by setting

$$[a] + [b] = [a + b], \qquad [a] \cdot [b] = [a \cdot b]$$

for each $a, b \in \mathbb{R}$. Which of these operations is well-defined?

(+) Say $[a] = [a']$, $[b] = [b']$

then $a' = a + x$ $\quad b' = b + y$ $\quad x, y \in \mathbb{Q}$

$\therefore a' + b' = a + b + \underbrace{x + y}_{\in \mathbb{Q}}$ $\quad \therefore [a' + b'] = [a + b]$ ✓

$(\cdot)$ $a'b' = (a + x)(b + y) = ab + \underbrace{xb + ya}_{\text{not necessarily in } \mathbb{Q}} + \underbrace{xy}_{\in \mathbb{Q}}$

(eg $a = \sqrt{2}, b = 0, x = 0, y = 1$)

$\therefore [a'b']$ might not equal $[ab]$

✗ not well-defined.