HW1 Solutions

§2.2 10

(a) $\sim$ on 2^S is defined by $A \sim B \Leftrightarrow \exists\, f: A \to B$ that is a 1-1 correspondence.

This is an equivalence relation.

Proof. $A \sim A$: Define $f: A \to A$, $x \mapsto x$. This is a 1-1 correspondence $\therefore A \sim A$ ✓

$A \sim B \Rightarrow B \sim A$: Say $A \sim B$. So there's $f: A \to B$ 1-1 correspondence. Let $f^{-1}: B \to A$ be its 2-sided inverse. Also a 1-1 correspondence $\therefore B \sim A$ ✓

$A \sim B, B \sim C \Rightarrow A \sim C$: Let $f: A \to B$ and $g: B \to C$ be 1-1 correspondences. Then $g \circ f: A \to C$ is a 1-1 correspondence. $\therefore A \sim C$ ✓

(b) $S = \{1,2,3,4\}$

$$2^S = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$

$\sim$-equivalence classes:

$\{\varnothing\}$

$\{\{1\}, \{2\}, \{3\}, \{4\}\}$

$\{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

$\{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$

$\{\{1,2,3,4\}\}$

FIVE

§2.2 <u>11</u>

(a) $\equiv$ is an equivalence relation

$u \equiv u \pmod W$: $u - u = 0 \in W$ (definition of subspace) ✓

$u \equiv v \pmod W \Rightarrow v \equiv u \pmod W$: $u - v \in W$ $\therefore v - u = -(u-v) \in W$ ✓

(subspaces are closed under scalars)

$u \equiv v \pmod W$, $v \equiv w \pmod W \Rightarrow u \equiv w \pmod W$: $u - v \in W$, $v - w \in W$

$\therefore u - w = (u-v) + (v-w) \in W$ ✓

(subspaces are closed under +)

(b) $u_1 \equiv v_1$ $u_2 \equiv v_2$

$\therefore u_1 - v_1 = w_1$, $u_2 - v_2 = w_2$ $w_1, w_2 \in W$

$\therefore r u_1 - r v_1 + s u_2 - s v_2 = r w_1 + s w_2 \in W$

$\therefore r u_1 + s u_2 \equiv r v_1 + s v_2 \pmod W$

(c) $[u]_W + [v]_W = [u+v]_W$ $r \cdot [u]_W = [ru]_W$.

These are <u>well-defined</u> (most important!)

eg say $u \equiv u'$, $v \equiv v' \pmod W$

then $u + v \equiv u' + v' \pmod W$ by (b)

$\therefore [u+v]_W = [u'+v']_W$ ✓

$U = \{ [u]_W \mid u \in V \}$ is a vector space (axiom check)

(d) $T: \mathbb{R} \to U$, $y \mapsto [(0,y)]_W$.

It's a linear transformation: $T(cy + c'y') = [(0, cy + c'y')]_W = c[(0,y)]_W + c'[(0,y')]_W$

$= c\,T(y) + c'\,T(y')$ ✓

It's 1-1: if $T(y) = T(y')$ then $[(0,y)]_W = [(0,y')]_W$ so $y \equiv y' \pmod W$ so $y = y'$ ✓

It's onto: $[(x,y)]_W = [(0,y)]_W = T(y)$ ✓

as $(x,0) \in W$

§5.2 <u>2</u> F a field $\varphi: F \to R$ hom.

To show φ is either 1–1 or zero,

suffices to show $\ker\varphi = \{0\}$ or F.

Suppose $\ker\varphi \neq \{0\}$. Then $\exists\ 0 \neq x \in \ker\varphi$.

Then $\langle x \rangle \subseteq \ker\varphi$

a unit, so $\langle x \rangle = F$ $\quad \therefore \ker\varphi = F$

§5.2 <u>4</u> $\varphi: \mathbb{C} \to \mathbb{C},\ z \mapsto \bar{z}$ is an automorphism.

<u>Proof</u> Additive is obvious, we checked multiplicative last term (write $z = x+iy$, $w = u+iv$ and expand both sides of $\varphi(zw) = \varphi(z)\varphi(w)$)

Finally, $\varphi^2 = Id$ so φ is its own two-sided inverse, hence bijective

§5.2 <u>5</u> Let $\varphi: \mathbb{Z} \to \mathbb{Z}$ be a homomorphism.

$\varphi(1) = 1$ $\quad \therefore \varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = 1+1 = 2$

$\therefore \varphi(3) = \varphi(2+1) = \varphi(2) + \varphi(1) = 2+1 = 3$

... etc ...

$\therefore \varphi(n) = n \quad \forall n > 0$

$\varphi(0) = 0$

$\varphi(-n) = -\varphi(n) \quad \therefore \varphi(n) = n \quad \forall n < 0$

$\therefore \varphi = Id$

§5.2 <u>7</u> $\varphi : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$

$$m+n\sqrt{2} \mapsto m-n\sqrt{2}$$

φ is an automorphism.

<u>Proof</u> Additive is tedious & easy.

Multiplicative:

$$\varphi((m+n\sqrt{2})(a+b\sqrt{2})) = \varphi(ma+2nb+(mb+na)\sqrt{2})$$
$$= ma+2nb-(mb+na)\sqrt{2}$$
$$= (m-n\sqrt{2})(a-b\sqrt{2})$$
$$= \varphi(m+n\sqrt{2})\,\varphi(a+b\sqrt{2}) \checkmark$$

Bijective:

$$\varphi^2 = Id$$

$\therefore \varphi$ is its own 2-sided inverse