Winter 2008

Elementary Abstract Algebra II Practise Final

Name:______________________________

1	2	3	4	5	6	7	8	TOT.

FINAL EXAM: 15:15–17:05 TUESDAY OF FINALS WEEK.
The real final will look roughly like this, probably slightly shorter questions, but similar topics.
Sections to revise: Well everything, of course, but pay special attention to chapters 5, 9 and 6.1–6.3. Try to read through all the *short* proofs from your notes as you know I expect you to be able to reproduce some of these. Try to learn definitions *precisely*.

1. Let $u = \sqrt[3]{2}$. Express $(1+u^2)^{-1}$ in the form $a + bu + cu^2$ for $a, b, c \in \mathbb{Q}$ in two different ways:
(a) by solving the equation $(a + bu + cu^2)(1 + u^2) = 1$;

(b) by using the Euclidean algorithm to write $GCD(x^2 + 1, x^3 - 2)$ as a linear combination of $x^2 + 1$ and $x^3 - 2$ then reducing modulo $x^3 - 2$.

2. Which of the following is a field? If it is a finite field, how many elements does it have?

(a) $\mathbb{Z}[x]/\langle x^2 + x + 1\rangle$.

(b) $\mathbb{Q}[x]/\langle x^2 + x + 3\rangle$.

(c) $\mathbb{Z}_5[x]/\langle x^2 + x + 1\rangle$.

(d) $\mathbb{Z}_5[x]/\langle x^2 + x + 3\rangle$.

3. Let R be a principal ideal domain (definition?). Let $a, b \in R$ be non-zero elements. Prove that a greatest common divisor of a and b (definition?) always exists. Show moreover that it can be expressed as a linear combination of a and b.

4. (a) Explain carefully why the polynomial $x^3 + 5x + 2$ is irreducible in $\mathbb{Z}_7[x]$.

(b) Write down a basis for $\mathbb{Z}_7[x]/\langle x^3 + 5x + 2\rangle$ as a $\mathbb{Z}_7$-vector space. What is $[\mathbb{Z}_7]x]/\langle x^3 + 5x + 2\rangle : \mathbb{Z}_7]$?

(c) Let $F = \mathbb{Z}_7[x]/\langle x^3 + 5x + 2\rangle$. Let $u = x + \langle x^3 + 5x + 2\rangle$. Express each of the three roots of the polynomial $x^3 + 5x + 2$ in the form $a + bu + cu^2$ for $a, b, c \in \mathbb{Z}_7$.

5. (a) What is an *isomorphism* $f : R \to S$ between two rings?

(b) By considering a suitable evaluation homomorphism, prove that $\mathbb{Q}[x]/\langle x^2 - 5\rangle$ is isomorphic to $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$.

(c) Is $\mathbb{R}[x]/\langle x^2 - 5\rangle$ isomorphic to $\mathbb{R}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{R}\}$?

6. Let $E \subseteq F$ be a finite field extension such that $[F : E]$ is prime. Suppose that $u \in F$ is algebraic over E.
(a) What are the possibilities for the degree of u over E?

(b) Suppose that a and b are complex numbers such that $[\mathbb{Q}(a) : \mathbb{Q}] = 5$ and $[\mathbb{Q}(b) : \mathbb{Q} = 7]$. Calculate $[\mathbb{Q}(a, b) : \mathbb{Q}]$.

(c) For a and b as in (b), write down a basis for $\mathbb{Q}(a, b)$ as a $\mathbb{Q}$-vector space.

7. Check you know the following definitions:
(a) Subfield.

(b) The minimal polynomial of an algebraic element $u \in F$ over a subfield E. (Explain *why* there is a unique such thing.)

(c) The field of fractions of an integral domain R. What do elements of the field of fractions of $F[x]$ look like? (This is usually denoted $F(x)$.)

(d) Unique factorization domain.

8. Take a look back at the two midterms this term! Solutions to them both are posted on the web!