Math 392 Homework 5 = Midterm review questions

- There is a midterm in class on Friday next week. It will be on everything we have done so far this term: basically, sections 3.2, 4.1 and 4.2 from the book. You must learn the basic definitions of things like homomorphisms, ideals, quotient rings, kernels, images, principal ideals, splitting fields, ... Other than that, the best way to prepare is to do this homework, then to look over all the examples you can find.

- Here are the problems for the homework. Problems on the midterm will be similar sorts of things, but not quite as hard so if you really understand these ones you'll do well.

1. Define the function $f : \mathbb{Z}[i] \to \mathbb{Z}_2$ by $f(x+iy) = [x+y]$ for all $x, y \in \mathbb{Z}$.
(i) Prove carefully that f is a ring homomorphism.
(ii) What is the *kernel* of f?
(iii) Is f any of: onto, 1–1, isomorphism?
(iv) Define $g : \mathbb{Z}[i] \to \mathbb{Z}_3$ by $f(x+iy) = [x+y]$ for all $x, y \in \mathbb{Z}$. Is g a ring homomorphism? Explain your answer carefully.

2. (i) What is an *ideal*?
(ii) Prove that the kernel of a ring homomorphism is an ideal.
(iii) Which of the following subsets of $\mathbb{Z}_{15}$ are ideals?
(a) $\{[0]\}$.
(b) $\{[1]\}$.
(c) $\{[0], [1], [2], \dots, [14]\}$.
(d) $\{[0], [5], [10]\}$.
(e) $\{[0], [4], [8], [12]\}$.
(iv) Let I be an ideal of a ring R. Prove that $I = R$ if and only if I contains a unit of R.

3. (i) I want to define a function $f : \mathbb{Z}_7 \to \mathbb{Z}_6$ by setting $f(n + (7)) = n + (6)$ for all $n \in \mathbb{Z}$. Explain why I cannot do this.
(ii) Instead, define $f : \mathbb{Z}_{42} \to \mathbb{Z}_6$ by setting $f(n + (42)) = n + (6)$ for all $n \in \mathbb{Z}$. Explain why I can do this.

4. Let $R = \mathbb{Z}_2[x]/(x^3 + x + 1)$.
(i) How many elements does R have? List them.
(ii) Is R an integral domain and/or a field? Explain.
(iii) Calculate the multiplicative inverse of the elements $x + (x^3 + x + 1)$ and $x^2 + (x^3 + x + 1)$ in R.

5. Suppose F is a field and $f(x) \in F[x]$.
(i) What is a *splitting field* for $f(x)$ over F?
(ii) Is $\mathbb{Q}[\sqrt{2} + \sqrt{3}] \subset \mathbb{R}$ a splitting field for the polynomial $(x^2 - 2)(x^2 - 3)$ over $\mathbb{Q}$? Explain.
(iii) Show that the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$ is $(x^4 - 10x^2 + 1)$.
(iii) Use the isomorphism theorem to prove that $\mathbb{Q}[x]/(x^4 - 10x^2 + 1) \cong \mathbb{Q}[\sqrt{2} + \sqrt{3}]$.
(iv) Construct a splitting field for the polynomial $(x^2 - 2)(x^2 - 3)$ over $\mathbb{Z}_7$.