392 Homework 2 solutions

- Exercises 3.2: 10. Decide whether $f(x) = x^3 - 2$ is irreducible in $\mathbb{Q}[\sqrt{2}][x]$.

  The roots are $\sqrt[3]{2}, \sqrt[3]{2}e^{\pm 2\pi i/3}$. *None of these belong to* $\mathbb{Q}[\sqrt{2}]$. *Since it is a cubic no roots implies its irreducible.*
- Exercises 4.2: 1 (a) Prove that the function $\phi : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}], a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an isomorphism. (b) Define $\phi : \mathbb{Q}[\sqrt{3}] \to \mathbb{Q}[\sqrt{7}]$ by $a + b\sqrt{3} \mapsto a + b\sqrt{7}$. Is ϕ an isomorphism? Is there any isomorphism?

  (a) It is obviously a 1–1 correspondence. Now we need to check that
  $$\phi(a + b\sqrt{2} + c + d\sqrt{2}) = \phi(a + b\sqrt{2}) + \phi(c + d\sqrt{2})$$
  which is obvious and
  $$\phi((a + b\sqrt{2})(c + d\sqrt{2})) = \phi(a + b\sqrt{2})\phi(c + d\sqrt{2}).$$
  For the last thing, the left hand side is $\phi((ac+2bd) + (ad+bc)\sqrt{2}) = (ac+2bd) - (ad+bc)\sqrt{2}$ *and the right hand side is* $(a-b\sqrt{2})(c-d\sqrt{2}) = (ac + 2bd) - (ad + bc)\sqrt{2}$ *so they are equal.*

  (b) No it is not an isomorphism because $\phi(\sqrt{3}\sqrt{3}) = \phi(3) = 3$ *which is not the same as* $\phi(\sqrt{3})\phi(\sqrt{3}) = \sqrt{7}\sqrt{7} = 7$. *There is NO isomorphism at all because* $\mathbb{Q}[\sqrt{3}]$ *contains no number which squares to 7 so it cannot be isomorphic to* $\mathbb{Q}[\sqrt{7}]$.
- Exercises 4.1: 9, 14(a)(c).

  I went over these ones or very similar ones in class so I'm not going to do them here.
- Exercises 3.3: 2(a)(c)(e), 4(a).

  2(a) Is $x^3 + 4x^2 - 3x + 5$ irreducible in $\mathbb{Q}[x]$?

  Its irreducible if and only if it has no rational roots which is if and only if it has no integer roots. Now sketch the graph: clearly positive for $x \geq 0$ *so no real roots at all. For* $x < 0$, *its* 11 *at* $x = -1$, 19 *at* $x = -2$, 23 *at* $x = -3$, 17 *at* $x = -4$ *and for* $x \leq -5$ *it is clearly negative. So no integer roots at all.*

  2(c) Is $x^3 + x^2 + x + 1$ irreducible in $\mathbb{Q}[x]$?

  NO because $x = -1$ *is a root.*

  2(e) Is $x^4 + x^2 - 6$ irreducible in $\mathbb{Q}[x]$?

  It factorizes as $(x^2 + 3)(x^2 - 2)$.

  4(a) Show $x^{200} - x^{41} + 4x + 1$ has no rational root.

  I want you to sketch the graph accurately enough to see that it doesn't cross the x*-axis at any integer. Hence it has no rational root since rational roots ARE integers for a monic polynomial in* $\mathbb{Z}[x]$.