Math 392 Homework 9 = Final review questions

This homework will not be graded. We will spend the Wednesday and Friday classes of dead week mainly on revision – then I will hand out solutions to these review questions and go over any problems.
Also on Wednesday of dead week I will talk about the term paper I am going to set for next term in lieu of a final exam!!!! I am telling you about this now because it is perfectly possible to work on this over spring break and get it well underway with before the spring term is under way.
Hints for the final: there will be 6 questions, each of the same length and difficulty questions on the midterm. The questions in the final will be on roughly the same topics as this review sheet. As usual, you must make sure you know:

- all the basic definitions (e.g. homomorphism, ideal, unit, irreducible element, quotient ring, principal ideals, algebraic numbers, transcendental numbers, cyclotomic polynomials);
- the main theorems (e.g. the isomorphism theorem, the Chinese remainder theorem, Euclidean algorithm);
- the main techniques (e.g. long division for integers and polynomials, finding GCDs, proving polynomials are irreducible, finding units in rings, constructing fields of order p^2);
- the main examples and constructions of rings (e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}[i]$, adjoining numbers to $\mathbb{Q}$, adjoining an indeterminate x to get a polynomial ring, direct products of rings, subrings, quotient rings, $\mathbb{Z}_n$ and $\mathbb{Z}_p$).

1. (i) Let R be a ring. What is an *irreducible element* $x \in R$?
(ii) Determine which of the following polynomials in $\mathbb{Z}_2[x]$ are irreducible, explaining your method carefully. For the ones that are reducible, factor them into irreducibles.
(a) $x^2 + 1$;
(b) $x^3 + 1$;
(c) $x^3 + x + 1$;
(d) $x^3 + x^2 + 1$;
(e) $x^3 + x^2 + x + 1$;
(iii) Which of the following Gaussian integers are irreducible? For the ones that are reducible, factor them into irreducibles.
(a) 1;
(b) 11;
(c) 17;
(d) $1 + i$;
(e) $10 + 10i$.

2. (i) If R is a ring and $a \in R$, define what the notation (a) means. Prove that (a) is an *ideal* of R.

(ii) Now let R be the ring $F[x]/(x^4+x^2+1)$. Suppose that

$$x^7 = a_3x^3 + a_2x^2 + a_1x + a_0$$

in R. Calculate the numbers a_0, a_1, a_2, a_3.
(iii) Factorize the polynomial x^4+x^2+1 into irreducibles, working in the ring $\mathbb{Z}_3[x]$.
(iv) Is the factor ring $\mathbb{Z}_3[x]/(x^4+x^2+1)$ a field? Explain.

3. Let m, n be coprime positive integers. The Chinese remainder theorem says that $\mathbb{Z}_{mn} \cong Z_m \times \mathbb{Z}_n$.
(i) List *all* the elements of the ring $\mathbb{Z}_2 \times \mathbb{Z}_3$.
(ii) Compute the *multiplication table* of $\mathbb{Z}_2 \times \mathbb{Z}_3$.
(iii) Write down the isomorphism between $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_6$ coming from the proof of the Chinese Remainder Theorem explicitly (i.e. pair up the elements of $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_6$ according to the 1–1 correspondence of the isomorphism).
(iv) Prove or disprove: $\mathbb{Z}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

4. (i) Define an *integral domain.*
(ii) Determine whether the ring $\mathbb{Z}[x]/(x^4 - 16)$ is an integral domain, explaining your answer carefully.

5. (i) State the Eisenstein criterion.
(ii) Determine which of the following polynomials in $\mathbb{Q}[x]$ are irreducible:
(a) $x^5 - 4x + 22$;
(b) $x^5 - 4x - 1$;
(c) $x^{11} - 6x^4 + 12x^3 + 36x - 6$.
Which of these are irreducible in $\mathbb{R}[x]$ instead?
(iii) Show that there are an infinite number of integers a such that $x^7 + 15x^2 - 30x + a$ is irreducible in $\mathbb{Q}[x]$.
(iv) Factor the polynomial $x^{24} - 1$ completely into irreducibles in $\mathbb{Q}[x]$.

6. Let F be a field. Define the map

$$\phi : F[x] \to F[x], \qquad f(x) \mapsto f(x+1).$$

So for example, $\phi(x) = x + 1, \phi(x^2) = (x+1)^2 = x^2 + 2x + 1$, etc... Prove carefully that ϕ is an isomorphism.

7. Find a non-zero zero divisor in the ring $\mathbb{Z}[i]/(5 + i)$.

8. Let R be a ring. Fix an element $a \in R$ and set

$$I_a = \{r \in R \mid r^2 a = 0\}.$$

Prove that I_a is an ideal of the ring R.