Math 391: Homework 1 (due W October 3)

Part I: reading. Begin by looking at Definition 5.1.2' in the text book. If there are words in that definition you don't understand, try to work them out by browsing through that section or other sections in the book. If that doesn't help – ask!

Then read pages 1–13.

Part II: the axioms for a ring.

Let R be a commutative ring. Prove the following things starting from the ring axioms using a bit of logic and NOTHING ELSE. This will feel a bit weird at first! You should try to write complete (mathematical) sentences with a subject verb and object. The aim in writing proofs is to make sure that what you write makes coherent sense to a fellow student in the class...

1. Let $a \in R$ be any element. Suppose that $b \in R$ is an element with the property that $a + b = 0$. Prove that $b = -a$. This shows: the special element of R called $-a$ is the *unique* element of R which when added to a gives zero. If you like: "additive inverses are unique".

2. Prove that $a \cdot 0 = 0$ for all $a \in R$.

3. Prove $a \cdot (-1) = -a$.

In class we said that an element $a \in R$ was *divisible by* $b \in R$ if $a = bc$ for some $c \in R$. In other words, b divides a or a is a multiple of b. Notation: $b|a$.

4. If a and b are divisible by c, prove that $a+b$, $a-b$ and ka are also divisible by c for any $k \in R$.

5. Which numbers in R are divisible by 0? Which numbers are divisible by 1?

A number $a \in R$ is called a *zero divisor* if $0 = ab$ for some $0 \neq b \in R$. This is not quite what you'd expect (every number divides zero but zero divisors are usually quite rare). Zero is always a zero divisor. If zero is the *only* zero divisor in R then R is called an *integral domain.*

6. Give an example of a commutative ring in which zero is the only zero divisor, i.e. an integral domain. Give an example of a commutative ring in which zero is not the only zero divisor.

Any number that divides 1 is called a *unit.* For example ± 1 are always units.

7. If $a \in R$ is a unit prove there exists a *unique* element $b \in R$ such that $ab = 1$. This element b is called a^{-1}, the multiplicative inverse of a.

8. If a and b are units prove that $(ab)^{-1} = a^{-1}b^{-1}$ and that $(-a)^{-1} = -(a^{-1})$.

Part III: questions from the book.

Exercises 1.1: 3,5,8,9,10,19,21.