

Fall 2007

Elementary Abstract Algebra I Practise Final

Name: _____

1	2	3	4	5	6	7	8	TOT.

FINAL EXAM: 15:15–17:05 THURSDAY OF FINALS WEEK.

The real final will look roughly like this, probably slightly shorter questions, but similar topics.

Sections to revise: Chapter 1 (all), Section 2.1, Chapter 4 (all), Section 5.1, Definition 5.2.1 (why is it automatic that $\phi(1) = 1$ if ϕ is one-to-one and onto?).

Then go over all homeworks and make sure you understand how to do them with hindsight!

Revising is a key part of learning mathematics: lots of things you didn't completely understand the first time round should be easier when you go back over it again!!

1. Let F be a field and $f(x) \in F[x]$ be an irreducible polynomial. Prove carefully that $f(x)|a(x)b(x)$ implies either $f(x)|a(x)$ or $f(x)|b(x)$.

2. (a) For which values of $a = 1, 2, 3, 4$ is $\mathbb{Z}_5[x]/\langle x^2 + x + a \rangle$ a field? Explain.

(b) For which values of $k = 2, 3, 4, 5$ is $\mathbb{Z}_k[x]/\langle x^2 + x + 1 \rangle$ a field? Explain (and be extra careful with $k = 4$).

3. Factor the polynomial $x^9 - 1$ completely into irreducibles in the ring $\mathbb{Q}[x]$. Use Eisenstein's criterion to prove that each of your irreducible factors really are irreducible.

4. (a) Let F be a field and $f(x) \in F[x]$ be a polynomial. Prove the *remainder theorem*:
“For $a \in F$, $(x - a)$ is a linear factor of $f(x)$ if and only if $f(a) = 0$.”

(b) Prove that $f(x) = x^3 + 3x^2 + 2x + 1$ is irreducible over \mathbb{Q} . (*Hint*: draw a rough sketch of the graph by plotting points at $x = -3, -2, -1$ and 0 and use a recent theorem about rational roots...)

5. (a) Prove for any $x \in \mathbb{Z}$ that $x^5 \equiv x \pmod{5}$.

(b) Find the multiplicative inverse of x in $\mathbb{Z}_5[x]/\langle x^{2007} + 2 \rangle$.

(c) Is the ring $\mathbb{Z}_5[x]/\langle x^{2007} + 2 \rangle$ a field?

6. (a) What is an *isomorphism* $f : R \rightarrow S$ between two rings?

(b) Prove that $\mathbb{Q}[x]/\langle x^2 - 5 \rangle$ is isomorphic to $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$.

(c) Is $\mathbb{R}[x]/\langle x^2 - 5 \rangle$ isomorphic to $\mathbb{R}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{R}\}$?

7. For each of the following, decide whether the function is *well-defined*, and if so determine whether it is *one-to-one* and/or *onto*.

(a) $f : \mathbb{C} \rightarrow \mathbb{C}, x + iy \mapsto 2y + i\left(\frac{\sqrt{3}}{2}x - y\right)$.

(b) $f : \mathbb{Z}_4 \rightarrow \mathbb{Z}_8, [x]_4 \mapsto [3x]_8$.

(c) $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}, (x, y) \mapsto x + iy$.

(d) $f : \mathbb{R} \rightarrow \mathbb{C}, x \mapsto x^2 + ix$.

(e) $f : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}, (x, y) \mapsto (x^2, x + y)$.

8. Here are five rings. Work out which of them are isomorphic to each other.

Explain!

$$R_1 = \mathbb{Z}_2[x]/\langle x^3 + x \rangle.$$

$$R_2 = \mathbb{Z}_8.$$

$$R_3 = \mathbb{Z}_2 \times \mathbb{Z}_4.$$

$$R_4 = \mathbb{Z}_2[x]/\langle x^2 + x + 1 \rangle.$$

$$R_5 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$