Practise!

Math 252H Midterm

Name:

Instructions

READ each question CAREFULLY.

Make sure you JUSTIFY your answers – that way, I can give some credit even for wrong answers.

Try to be as CLEAR and SUCCINCT as possible.

Answer ALL questions. Each question is worth TEN points.

Question	Points
1	
2	
3	
4	
5	
Total	

FTC I f int'ble on $[a,b]$, cts at $c \in [a,b]$ $\Rightarrow F(x) = \int_a^x f(t)\,dt$ is diff'ble at c, $F'(c) = f(c)$

FTC II f int'ble on $[a,b]$, g any anti-derivative $\Rightarrow \int_a^b f(x)\,dx = g(b) - g(a)$

1. State the fundamental theorem of calculus as accurately as you can.

2. In this question, f denotes the function $f(x) = x^5 + x^3 + 1$.

(i) Prove that f is increasing.

$f'(x) = 5x^4 + 3x^2$

On $(0,\infty)$ this is +ve, so f is increasing on $(0,\infty)$

On $(-\infty,0)$ this is +ve, so f is increasing on $(-\infty,0)$

Also $f(0)=0$ $f(x)>0$ for $x>0$, $f(x)<0$ for $x<0$

$\therefore$ Can patch to see f is increasing on all of $\mathbb{R}$

(ii) Calculate $(f^{-1})'(3)$.

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{(5x^4+3x^2)\big|_{x=1}} = \frac{1}{8}$$

$f(1)=3$

$\therefore f^{-1}(3) = 1$

(iii) Is f^{-1} differentiable everywhere? Explain.

No, $f'(0)=0$

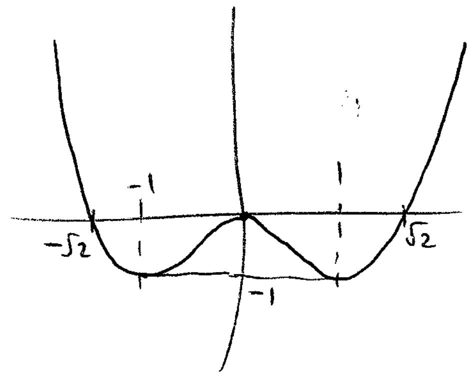
$\therefore$ f^{-1} is not differentiable at x where $f^{-1}(x)=0$, ie $x=1$.

3. Sketch the graphs of the following functions, clearly labelling the important features like critical points, asymptotes, axis intercepts, ...

(i) $f(x) = x^4 - 2x^2$.

$f'(x) = 4x^3 - 4x = 0$

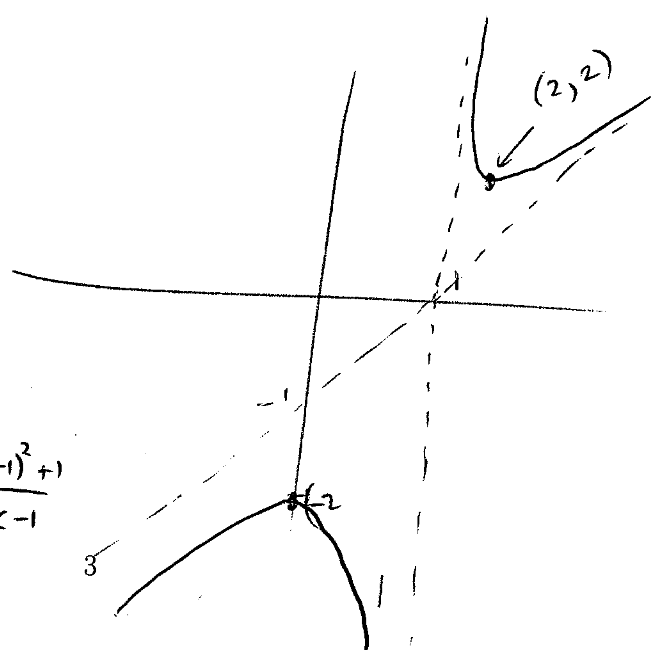
$x = 0, \pm 1$



(ii) $f(x) = \dfrac{x^2 - 2x + 2}{x - 1}$. $= \dfrac{(x-1)^2 + 1}{x-1}$

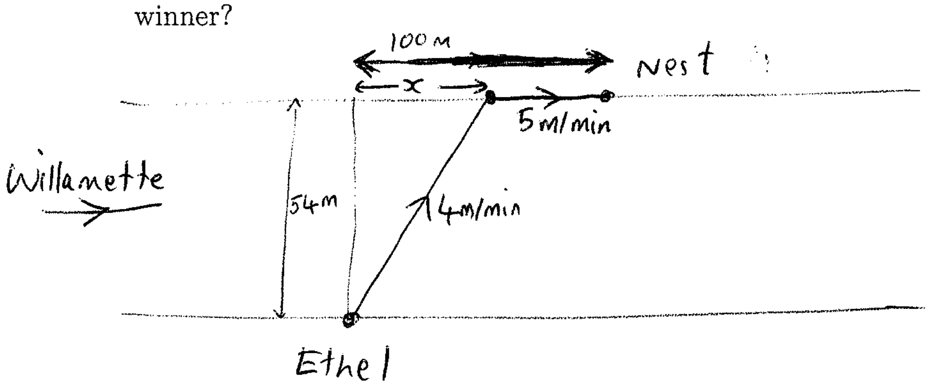
$= x - 1 + \dfrac{1}{x-1}$

$f'(x) = 1 - \dfrac{1}{(x-1)^2} = 0 \quad x = 0, 2$



4. Ethel the frog wants to get to her nest located 100 meters downstream from her but on the opposite side of the Willamette river. She can swim at 4 m/min (regardless of the current which you should ignore!) and hop at 5 m/min. She decides to swim in a straight line across to the other side, then to hop along the bank the remaining distance. The Willamette river is 54 metres wide, the time now is 6.32pm and the FrogoLotto results will be announced at 7pm exactly. Can she get home in time to see if she is the winner?

need time ≤ 28 min to get home



Endpoints

$T(0) = 33.5$ ×

$T(100) = 28.4$ ×

$$T(x) = \frac{100-x}{5} + \frac{\sqrt{x^2+54^2}}{4} \quad \text{on } [0,100]$$

$$T'(x) = -\frac{1}{5} + \frac{x}{4\sqrt{x^2+54^2}} = 0$$

$$5x = 4\sqrt{x^2+54^2}$$

$$25x^2 = 16x^2 + 216^2$$

$$9x^2 = 216^2$$

$$3x = 216 \qquad x = \frac{216}{3}$$

Critical point

$$T\left(\frac{216}{3}\right) = T(72) = 28.1 \times$$

$\boxed{\text{Nope}}$

5. Let $f(x) = x$.
(i) Let $\mathcal{P}$ be the partition $0 < 1/n < 2/n < \cdots < (n-1)/n < 1$ of the interval $[0,1]$. Calculate $U(f,\mathcal{P}) - L(f,\mathcal{P})$ in terms of n.

$$U(f,P) = \sum_{i=1}^{n} \frac{1}{n}\cdot\frac{i}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{n(n+1)}{2n^2} = \boxed{\frac{n+1}{2n}}$$

$$L(f,P) = \sum_{i=1}^{n} \frac{1}{n}\cdot\frac{i-1}{n} = \sum_{i=1}^{n}\frac{1}{n}\cdot\frac{i}{n} - \sum_{i=1}^{n}\frac{1}{n^2} = U(f,P) - \frac{1}{n}$$

$$\therefore U(f,P) - L(f,P) = \boxed{\frac{1}{n}}$$

(ii) Explain why your answer to (a) proves that f is integrable on $[0,1]$.

Well by the theorem: f is int'ble iff $\forall \varepsilon > 0 \ \exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

(iii) Give an example of a bounded function on $[0,1]$ which is *not* integrable, explaining your answer briefly.

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

For this $U(f,P) = 1 \quad \forall P$

$L(f,P) = 0 \quad \forall P$

$\therefore \underline{\int} f = 0, \quad \overline{\int} f = 1$

$\therefore$ Not integrable.