Homework 4 (some) solutions

1. (a) State the Mean Value Theorem (carefully!)

(b) Prove that if $f'(x) < 0$ on an interval then f is decreasing on that interval.

(c) Suppose that f is continuous on an interval $[a, b]$, and $c \in (a, b)$. Prove that if $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all x in (c, b) then c is a local minimum of f.

Solution. (a) If f is cts on $[a, b]$ and diff'ble on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) See the proof of chapter 11, Corollary 3.

(c) Suppose for a contradiction that c is not a local minimum of f. Choose $\delta > 0$ such that $(c - \delta, c + \delta)$ is contained in the interval (a, b). Since c is not a local minimum, there must exist $x \in (c - \delta, c + \delta)$ such that $f(x) < f(c)$.

Let me now just discuss the case $x > c$, the case $x < c$ being similar. Take $f(x) < d < f(c)$. Since f is continuous on $[c, x]$, there exists $y \in (c, x)$ with $f(y) = d$. But then we have $c < y < x$ with $f(x) < f(y)$. This contradicts (b), since that implies that f is increasing on (c, b).

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2. Do problem 16 of chapter 11.

Solution. Let's say the angle of the sector going horizontally to the left and up to A is θ. Then the length $A + B$ is $a \sin\theta + a + a\cos\theta$. We need to maximize this for $\theta \in [0, \pi]$. At 0 we get $2a$, at π we get 0. Then for critical points we need $\cos\theta = \sin\theta$ so $\theta = \pi/4$, when the length is $a(1 + \sqrt{2})$. That is the maximum length...

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3. Consider the function $f(x) = 1/(1 + x^2)$ on the interval $[0, \infty)$.

(a) Prove that f is DECREASING on this interval. (Sorry, the question was wrong – did you notice?)

(b) What is the domain of f^{-1}?

Solution. (a) We show that $f'(x) < 0$ always. We have $f'(x) = -\frac{2x}{(1+x^2)^2}$. For $x > 0$, this is always negative.

(b) At $x = 0$, $f(x) = 1$ and as $x \to \infty$, $f(x) \to 0$. So the outputs of the function are always between 0 and 1. This is the domain of f^{-1}: $(0, 1]$.

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4. Prove that if $f(x)$ is increasing then so is $f^{-1}(x)$.

Solution. We need to show that if $f(a) < f(b)$ then $f^{-1}(f(a)) < f^{-1}(f(a))$ (because everything in the domain of f^{-1} looks like $f(a)$ for some a). Let us assume for a contradiction that $f(a) < f(b)$ but $f^{-1}(f(a)) \geq f^{-1}(f(a))$. Then $a \geq b$.

Case one: $a = b$. Then $f(a) = f(b)$, contrary to assumption.

Case two: $a > b$. Then $f(a) > f(b)$ as f is increasing, contrary to assumption.

Either way, we got the desired contradiction.

5. Find the derivatives of the following:
   (a) $f(x) = \sin(x + \sin(x + \sin(x)))$.
   (b) $f(x) = (x^2 + (x^2 + (x^2 + 1)^2)^2)^2$.
   Solution.
   (a) $f'(x) = \cos(x + \sin(x + \sin(x)))[1 + \cos(x + \sin(x))(1 + \cos(x))]$.
   (b) $f'(x) = 2(x^2+(x^2+(x^2+1)^2)^2)[2x+2(x^2+(x^2+1)^2)(2x+4x(x^2+1))]$.

6. Chapter 13, question 1.
   Solution. Let $P_n = \{t_0 < t_1 < \cdots < t_n\}$ where $t_i = ib/n$.
   Let's calculate:

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = \sum_{i=1}^{n} (ib/n)^3 \, b/n \\ &= (b/n)^4 \sum_{i=1}^{n} i^3 = (b/n)^4 1/4n^2(n+1)^2 \\ &= b^4/4 \left(1 + 2/n + 1/(n^2)\right). \end{aligned}$$

and similarly,

$$L(f, P_n) = b^4/4 \left(1 - 2/n + 1/(n^2)\right)$$

Now using this and thinking about n very large gives that

$$\sup\{L(f, P_n) \mid n \geq 1\} = b^4/4, \qquad \inf\{U(f, P_n) \mid n \geq 1\} = b^4/4.$$

But f is integrable, so

$$\begin{aligned} b^4/4 &= \sup\{L(f, P_n) \mid n \geq 1\} \leq \sup\{L(f, P) \mid \forall P\} = \inf\{U(f, P) \mid \forall P\} \\ &\leq \inf\{U(f, P_n) \mid n \geq 1\} = b^4/4. \end{aligned}$$

Hence equality holds everywhere and we've computed the integral to be $b^4/4$.