Winter 2002

Honors Calculus II Final Exam

Name:____________________________

1	2	3	4	5	6	7	8	TOT.

Answer ALL questions. Show all your work! If you finish early, TRIPLE CHECK your answers!!

(1) (a) [2 points] Let a be a positive real number. Write down a formula for a^x in terms of the exponential function e^x.

$$a^x = e^{x \log a}$$

(b) [2 points] If $f(x) = 10^x$, calculate $f'(x)$.

$$f(x) = e^{x \log 10}$$

$$\therefore f'(x) = e^{x \log 10} \cdot \log 10 = 10^x \cdot \log 10$$

(c) [3 points] If $f(x) = x^x$, calculate $f'(x)$.

$$f(x) = e^{x \log x}$$

$$f'(x) = e^{x \log x} (\log x + 1) = x^x (\log x + 1)$$

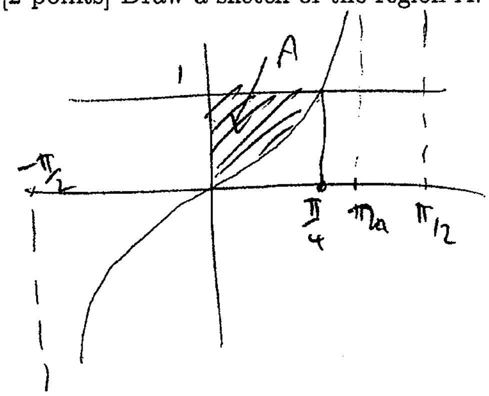
(d) [3 points] If $f(x) = \log(x^x)$, calculate $f'(x)$.

$$f(x) = x \log x$$

$$\therefore f'(x) = \log x + 1$$

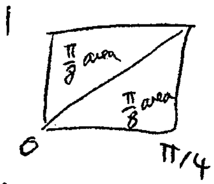
(2) Let A be the region lying to the right of the y-axis, below the horizontal line $y = 1$ and above the graph of the function $f(x) = \tan(x)$.

(a) [2 points] Draw a sketch of the region A.



$\tan x = 1$ when $x = \frac{\pi}{4}$

(b) [2 points] Explain briefly using your picture why you should expect the area of the region A to be between $\pi/8$ and $\pi/4$.



Its more than half the rectangle which has area $\frac{\pi}{4}$, but less than all...

Check its between $\frac{\pi}{8}$ and $\frac{\pi}{4}$!

(c) [4 points] Use the substitution $u = \cos x$ to calculate

$$\int \frac{\sin x}{\cos x}\,dx.$$

$du = -\sin x\,dx$

$$-\int \frac{du}{u} = -\log u = -\log\cos x$$

(d) [4 points] Now use integration to calculate the area of the region A exactly.

$$\frac{\pi}{4} - \int_0^{\pi/4} \tan x\,dx = \frac{\pi}{4} + \Big[\ln\cos x\Big]_0^{\pi/4}$$

$$= \frac{\pi}{4} + \log\frac{\sqrt{2}}{2}$$

(or $\frac{\pi}{4} - \log\sqrt{2}$)

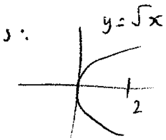
(3) (a) [4 points] Calculate the length of the curve $y = x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$.

$$\int_0^1 \sqrt{1+\left(\tfrac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^1 \sqrt{1+\tfrac{9}{4}x}\,dx$$

$$= \left[\tfrac{2}{3}\cdot\tfrac{4}{9}\left(1+\tfrac{9}{4}x\right)^{3/2}\right]_0^1 = \frac{8}{27}\left(\left(\tfrac{13}{4}\right)^{3/2}-1\right)$$

(b) [4 points] Calculate the volume of the solid obtained when the parabola $y = x^2$ is rotated around the y-axis, between $y = 0$ and $y = 2$.

Same as:



$$\pi\int_0^2 x\,dx = \frac{\pi}{2}\left[x^2\right]_0^2 = 2\pi$$

(c) [4 points] Calculate the curved surface area of the solid obtained by rotating the curve $y = \sin x$ from $x = 0$ to $x = \pi$ around the x-axis.

$$2\pi\int_0^\pi \sin x\sqrt{1+\cos^2 x}\,dx \quad \cancel{2\pi\left[\tfrac{2}{3}\cdot 2\left(1+\cos^2 x\right)^{3/2}\right]}$$

Let $u=\cos x$ $\qquad 2\pi\int_{-1}^{1}\sqrt{1+u^2}\,du = 4\pi\int_0^1\sqrt{1+u^2}\,du$

Let $u=\sinh x$

$$4\pi\int_0^{\log(1+\sqrt{2})}\cosh x\sqrt{1+\sinh^2 x}\,dx$$

$$= 4\pi\int_0^{\log(1+\sqrt{2})}\cosh^2 x\,dx = 4\pi\int_0^{\frac{1}{2}(e-\frac{1}{e})}\frac{1}{4}\left(e^{2x}+2+e^{-2x}\right)dx$$

$$= \pi\left[\tfrac{1}{2}e^{2x}+2x-\tfrac{1}{2}e^{-2x}\right]_0^{\log(1+\sqrt{2})}$$

$$= \pi\left[\tfrac{1}{2}e^{\log(1+\sqrt{2})^2}+2\log(1+\sqrt{2})-\tfrac{1}{2}e^{\log(1+\sqrt{2})^{-2}}\right]$$

$$= \pi\left(\tfrac{1}{2}(1+\sqrt{2})^2-\tfrac{1}{2}(1+\sqrt{2})^{-2}+2\log(1+\sqrt{2})\right)$$

$\sinh x = 1$

$\frac{1}{2}(e^x - e^{-x}) = 1$

$e^{2x} - 2e^x - 1 = 0$

$e^x = 1 \pm \sqrt{2}$

$x = \log(1+\sqrt{2})$

(3) (a) [2 points] Give the definition of the function $\log x$ for $x > 0$.

$$\log x = \int_1^x \frac{1}{t}\,dt$$

(b) [4 points] Using a substitution of the form $u = f(x)$ for a suitable f, prove that

$$\int_{x=a}^{x=ab} \frac{1}{x}\,dx = \int_{u=1}^{u=b} \frac{1}{u}\,du.$$

Show your working!

$$\int_a^{ab} \frac{1}{x}\,dx \longrightarrow \int_1^b \frac{a}{\cancel{x}u \cdot a}\,du = \int_1^b \frac{1}{u}\,du$$

Let ~~tx~~ $u = \frac{x}{a}$

$du = \frac{1}{a}\,dx$

(c) [4 points] Deduce that $\log(ab) = \log a + \log b$.

$$\log(ab) = \int_1^{ab} \frac{1}{x}\,dx = \int_1^a \frac{1}{x}\,dx + \int_a^{ab} \frac{1}{x}\,dx$$

(b)

$$= \int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx$$

$$= \log a + \log b$$

(4) Compute the following integrals:

(a) [2 points] $\displaystyle\int_0^{\pi/4} \sin(2x)\,dx.$

$$\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/4} = --\frac{1}{2} = \frac{1}{2}$$

(b) [2 points] $\displaystyle\int xe^{x^2}\,dx.$

$$\frac{1}{2}e^{x^2}$$

(c) [2 points] $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx.$

$$\left[\frac{1}{2}\log(1+x^2)\right]_0^1 = \frac{1}{2}\log 2 = \log\sqrt{2}$$

(d) [4 points] $\displaystyle\int_0^1 \frac{1}{x^2}\,dx.$

$$\lim_{\varepsilon\to 0^+}\left[-\frac{1}{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+}\left(\frac{1}{\varepsilon}-1\right) = \infty$$

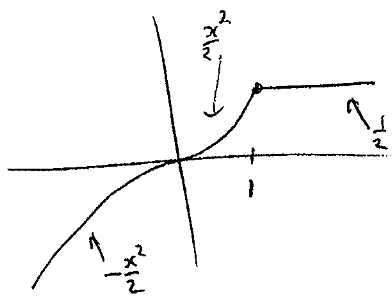
(5) (a) [3 points] State (carefully!) the fundamental theorem of calculus.

If f is integrable on $[a,b]$ and continuous at $c \in (a,b)$, then $F(x) = \int_a^x f(t)\,dt$ is differentiable at c with $F'(c) = f(c)$.

(b) [4 points] Let $f(x)$ be the function defined by

$$f(x) = \begin{cases} |x| & x \leq 1 \\ 0 & x > 1. \end{cases}$$

Let $F(x) = \int_0^x f(t)\,dt$. Sketch a graph of the function $F(x)$.



(c) [3 points] Determine exactly for which points x we have that $F'(x) = f(x)$.

Its true except possibly at $x = 1$ by FTC.

Its false at $x = 1$ by inspection of the graph.

(6) [8 points] Find the minimum value taken by the function $f(x) = 4x + \frac{1}{x}$ on the interval $(0, 1)$.

Not at endpoints clearly.

$\therefore$ At critical points.

$f'(x) = 4 - \frac{1}{x^2} = 0$ $\quad$ ~~$x = \pm\frac{1}{2}$~~ $\quad x = \frac{1}{2}$

$f(x) = 2 + 2 = 4$

(7) In this question, you may assume the formula $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

(a) [2 points] Prove using this formula that $\cos 2x = \cos^2 x - \sin^2 x$ and that $1 = \cos^2 x + \sin^2 x$.

Set $y = x$ to get $\cos 2x = \cos^2 x - \sin^2 x$

Set $y = -x$ to get $\cos 0 = 1 = \cos x \cos(-x) - \sin x \sin(-x)$

$= \cos^2 x + \sin^2 x$

(b) [2 points] Hence show that $\cos 2x = 2\cos^2 x - 1$.

$\cos 2x = \cos^2 x - (1 - \cos^2 x)$

$= 2\cos^2 x - 1$

(c) [3 points] Using your answer to (b), compute

$$\int \cos^2 x \, dx.$$

$= \int \frac{\cos 2x + 1}{2} dx = \frac{1}{4} \sin 2x + \frac{1}{2} x$

(d) [3 points] By making the substitution $x = \sin u$, compute

$$\int \sqrt{1-x^2}\, dx.$$

$dx = \cos u \, du$

$\rightarrow \int \sqrt{1-\sin^2 u} \cos u \, du = \int \cos^2 u \, du = \frac{1}{4} \sin 2u + \frac{1}{4} u$

$= \frac{1}{2} \sin u \cos u + \frac{1}{4} u$

$= \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{4} \arcsin x$

(8) This question is concerned with two special functions denoted $s(x)$ and $c(x)$. The only thing I am going to tell you about these functions is the following:

$\frac{1}{2}(e^{x}-e^{-x})$

Theorem. *If $f(x)$ is any twice differentiable function such that*

$$f''(x) = f(x),$$

then $f(x) = f(0)c(x) + f'(0)s(x)$.

(a) [2 points] The function $f(x) = e^x$ is a twice differentiable function with $f''(x) = f(x)$. Apply the Theorem to show that $e^x = c(x) + s(x)$. Show similarly that $e^{-x} = c(x) - s(x)$.

$e^0 = 1$, so $e^x = c(x) + s(x)$ ✓

If $f(x) = e^x$, $f(0) = f'(0) = 1$

If $f(x) = e^{-x}$, $f(0) = 1$, $f'(0) = -1$ so $e^{-x} = c(x) - s(x)$ ✓

(b) [3 points] Use your answer to (a) to prove that $c(x) = \frac{e^x+e^{-x}}{2}$ and $s(x) = \frac{e^x-e^{-x}}{2}$.

$\frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\big((c(x)+s(x)) + (c(x)-s(x))\big) = c(x)$ ✓

$\frac{1}{2}(e^x - e^{-x}) =$ " — " $= s(x)$ ✓

(c) [2 points] Deduce that $c'(x) = s(x)$ and $s'(x) = c(x)$.

$c'(x) = \frac{d}{dx}\left(\frac{1}{2}(e^x + e^{-x})\right) = \frac{1}{2}(e^x - e^{-x})$ by chain rule $= s(x)$ ✓

$s'(x) = \frac{1}{2}(e^x + e^{-x})$ by chain rule $= c(x)$ ✓

(d) [3 points] Let $f(x) = s(x + y)$, where y is fixed. Show that $f''(x) = f(x)$. Hence, prove that $s(x + y) = s(x)c(y) + s(y)c(x)$. Find a similar formula for $c(x + y)$.

$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(c(x+y)) = s(x+y) = f(x)$ ✓

$f(0) = s(y)$ $\quad f'(0) = c(y)$ $\overset{\text{Theorem}}{\Longrightarrow}$ $s(x+y) = c(x)s(y) + s(x)c(y)$ ✓

Similarly, $c(x+y) = c(x)c(y) + s(x)s(y)$