

Winter 2002

Honors Calculus II Final Exam

Name: _____

1	2	3	4	5	6	7	8	9	TOT.

Answer ALL questions. Show all your work! If you finish early, TRIPLE CHECK your answers!!

(1) (a) [2 points] Let a be a positive real number. Write down a formula for a^x in terms of the exponential function e^x .

(b) [2 points] If $f(x) = 10^x$, calculate $f'(x)$.

(c) [3 points] If $f(x) = x^x$, calculate $f'(x)$.

(d) [3 points] If $f(x) = \log(x^x)$, calculate $f'(x)$.

(2) Let A be the region lying to the right of the y -axis, below the horizontal line $y = 1$ and above the graph of the function $f(x) = \tan(x)$.

(a) [2 points] Draw a sketch of the region A .

(b) [2 points] Explain briefly using your picture why you should expect the area of the region A to be between $\pi/8$ and $\pi/4$.

(c) [4 points] Use the substitution $u = \cos x$ to calculate

$$\int \frac{\sin x}{\cos x} dx.$$

(d) [4 points] Now use integration to calculate the area of the region A exactly.

(3) (a) [4 points] Calculate the length of the curve $y = x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$.

(b) [4 points] Calculate the volume of the solid obtained when the parabola $y = x^2$ is rotated around the y -axis, between $y = 0$ and $y = 2$.

(c) [4 points] Calculate the curved surface area of the solid obtained by rotating the curve $y = \sin x$ from $x = 0$ to $x = \pi$ around the x -axis.

(4) (a) [2 points] Give the definition of the function $\log x$ for $x > 0$.

(b) [4 points] Using a substitution of the form $u = f(x)$ for a suitable f , prove that

$$\int_{x=a}^{x=ab} \frac{1}{x} dx = \int_{u=1}^{u=b} \frac{1}{u} du.$$

Show your working!

(c) [4 points] Deduce that $\log(ab) = \log a + \log b$.

(5) Compute the following integrals:

(a) [2 points] $\int_0^{\pi/4} \sin(2x) dx.$

(b) [2 points] $\int x e^{x^2} dx.$

(c) [2 points] $\int_0^1 \frac{x}{1+x^2} dx.$

(d) [4 points] $\int_0^1 \frac{1}{x^2} dx.$

(6) (a) [3 points] State (carefully!) the fundamental theorem of calculus.

(b) [4 points] Let $f(x)$ be the function defined by

$$f(x) = \begin{cases} |x| & x \leq 1 \\ 0 & x > 1. \end{cases}$$

Let $F(x) = \int_0^x f(t) dt$. Sketch a graph of the function $F(x)$.

(c) [3 points] Determine exactly for which points x we have that $F'(x) = f(x)$.

- (7) [8 points] Find the minimum value taken by the function $f(x) = 4x + \frac{1}{x}$ on the interval $(0, 1)$.

(8) In this question, you may assume the formula $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

(a) [2 points] Prove using this formula that $\cos 2x = \cos^2 x - \sin^2 x$ and that $1 = \cos^2 x + \sin^2 x$.

(b) [2 points] Hence show that $\cos 2x = 2 \cos^2 x - 1$.

(c) [3 points] Using your answer to (b), compute

$$\int \cos^2 x \, dx.$$

(d) [3 points] By making the substitution $x = \sin u$, compute

$$\int \sqrt{1-x^2} \, dx.$$

- (9) This question is concerned with two special functions denoted $s(x)$ and $c(x)$. The only thing I am going to tell you about these functions is the following:

Theorem. *If $f(x)$ is any twice differentiable function such that*

$$f''(x) = f(x),$$

then $f(x) = f(0)c(x) + f'(0)s(x)$.

(a) [2 points] The function $f(x) = e^x$ is a twice differentiable function with $f''(x) = f(x)$. Apply the Theorem to show that $e^x = c(x) + s(x)$. Show similarly that $e^{-x} = c(x) - s(x)$.

(b) [3 points] Use your answer to (a) to prove that $c(x) = \frac{e^x + e^{-x}}{2}$ and $s(x) = \frac{e^x - e^{-x}}{2}$.

(c) [2 points] Deduce that $c'(x) = s(x)$ and $s'(x) = c(x)$.

(d) [3 points] Let $f(x) = s(x + y)$, where y is fixed. Show that $f''(x) = f(x)$. Hence, prove that $s(x + y) = s(x)c(y) + s(y)c(x)$. Find a similar formula for $c(x + y)$.